

Dissipationless mechanism of skyrmion Hall current in double-exchange ferromagnets

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We revisit a theory of skyrmion transport in ferromagnets. On a basis of an effective U(1) gauge theory for spin-chirality fluctuations in double-exchange ferromagnets, we derive an expression for the velocity of a skyrmion core driven by the dc electric field. We find that mutual feedback effects between conduction electrons and localized spins give rise to Chern-Simons terms, suggesting a dissipationless mechanism for the skyrmion Hall current. A conventional description of the current-induced skyrmion motion, appearing through the spin transfer torque and scattering events, is reproduced in a certain limit of our description, where the Chern-Simons terms are not fully incorporated. Our theory is applicable to not only metallic but also insulating systems, where the purely topological and dissipationless skyrmion Hall current can be induced in the presence of an energy gap.

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I. INTRODUCTION

Soliton dynamics in the presence of fermions has been one of the most fundamental issues in various fields of physics. It plays a central role in domain-wall dynamics in conducting polymers [1] and vortex dynamics in superconductors in the field of condensed-matter physics [2]. It is also relevant to a confinement and baryon dynamics in high-energy physics [3]. Skyrmions [4] were shown to appear as elementary excitations in the quantum Hall system [5], and the interplay between skyrmions and spin-wave excitations was also studied [6]. Actually, skyrmion excitations have been observed in the cold atom system [7], and their crystallization has also been observed in both two and three dimensional helical magnets [8, 9].

Advances in spintronics have promoted intensive and extensive studies on domain-wall dynamics in magnetic systems for its application to the magnetic memory device [10, 11]. In particular, spin-polarized electric currents allow for an efficient control of domain-wall dynamics because of the spin torque transfer. A conventional theoretical approach to this phenomenon is based on the Landau-Lifshitz-Gilbert equation, that is, the equation of motion for a single spin. In this formalism the so-called Gilbert damping term is introduced in a phenomenological manner [11] to take account of the dissipation in the spin dynamics, caused by the coupling to the conduction electrons and the relativistic spin-orbit coupling. It reveals that the spin current in the itinerant ferromagnet drives a domain-wall motion to the longitudinal direction [11]. The spin dynamics in nanoscale magnets has also been studied by introducing collective coordinates [12], such as positions of domain wall and magnetic vortices. Then, it has been argued that the spin current generates half magnetic vortices and/or anti-vortices which can exhibit a nontrivial motion, including the transverse motion in the presence of the Gilbert damping. In these previous

approaches spin currents are given, thus feedback effects of the spin dynamics on the electron dynamics have not been considered.

In this paper, we develop a theory of skyrmion transport in double-exchange ferromagnets, where conduction electrons interact with localized spins via the Hund's-rule coupling. An essential aspect of our study lies in a mutual feedback effect between electron spin currents and skyrmion currents, resulting in the dissipationless skyrmion Hall current of the topological origin, which should be distinguished from the dissipative skyrmion Hall current replying on the scattering events [11]. Actually, a coupling between the spin current and a skyrmion motion has also been discussed in Refs. [13–16], where topological magnetic textures generate electric currents via the Berry-phase induced electro-motive force, which produces feedback on the magnetization dynamics via a spin-transfer torque. In these pioneering works, however, they considered the dissipative mechanism of the skyrmion Hall current but not the dissipationless mechanism.

In Sec. II, resorting to the CP^1 representation for the localized spin, we derive an effective field theory for itinerant electrons and bosonic spinons, which interact via gauge fluctuations representing the spin chirality. Based on this effective field theory for the strong Hund's-rule coupling limit [11], we derive the Maxwell-Chern-Simons equations for both internal U(1) gauge fluctuations and external electromagnetic fields. The emergence of Chern-Simons terms is ascribed to mutual feedbacks between itinerant electrons and skyrmions. Focusing on the center-of-mass motion of the skyrmion, we obtain the velocity of the skyrmion core in terms of the dc electric field. The Chern-Simons terms induce the dissipationless skyrmion Hall current normal to the applied electric field. For comparison with previous theories, we show in Sec. III that our U(1) gauge-theory formulation for

skyrmion dynamics reproduces the dissipative skyrmion Hall current in the Landau-Lifshitz-Gilbert-equation approach.

In fact, the topological contribution of the dissipationless skyrmion Hall current scales with the skyrmion density, and thus vanishes in the thermodynamic limit when a single skyrmion is considered. However, we find that the Rashba spin-orbit coupling for conduction electrons produces a finite skyrmion Hall current in the thermodynamic limit, allowing for an observation of this intrinsic skyrmion Hall effect. This could be realized at the surface of three-dimensional topological insulators [17] when a ferromagnet is deposited. Recently, a dissipationless mechanism for magnetization switching was proposed in the topological surface state, where the Chern-Simons

term plays an essential role [18].

II. U(1) GAUGE THEORY FOR SINGLE SKYRMION DYNAMICS

A. An effective Maxwell-Chern-Simons Lagrangian

We start from an effective U(1) gauge-field formulation of a two-dimensional double exchange model with the Rashba spin-orbit coupling, where itinerant electrons interact with localized spins via the ferromagnetic Hund's-rule coupling J_H . This is described with the following partition function Z and the Lagrangian density \mathcal{L} ;

$$\begin{aligned} Z &= \int D\psi_\sigma D z_\sigma D a_\mu \delta(|z_\sigma|^2 - 1) \delta(\partial_\tau a_\tau) e^{-\int_0^\beta d\tau \int d^2r \mathcal{L}_{eff}}, \quad \mathcal{L} = \mathcal{L}_B + \mathcal{L}_\psi + \mathcal{L}_z, \\ \mathcal{L}_B &= 2iS a_\tau, \quad \mathcal{L}_\psi = \psi_\sigma^\dagger (\partial_\tau - \mu_\tau - J_H S \sigma - i\sigma a_\tau - iA_\tau) \psi_\sigma + t |[\partial_\mathbf{r} - i\sigma a_\mathbf{r} - iA_\mathbf{r} - i\sigma(\lambda_{so}/t) z_\alpha^\dagger \sigma_{\alpha\beta}^\mathbf{r} z_\beta] \psi_\sigma|^2, \\ \mathcal{L}_z &= \rho_s z_\sigma^\dagger (\partial_\tau - i a_\tau) z_\sigma + t \rho_s |[\partial_\mathbf{r} - i a_\mathbf{r} - i(\lambda_{so}/t) z_\alpha^\dagger \sigma_{\alpha\beta}^\mathbf{r} z_\beta] z_\sigma|^2. \end{aligned} \quad (1)$$

This effective field theory can be derived along the concept of Ref. [19]. The derivation is given in Ref. [20] as well as in Appendix A. The underlying mechanism to justify this formulation is that the spin dynamics of itinerant electrons instantaneously follows that of localized spins in the strong Hund's-rule coupling limit as far as the dynamics of localized spins are much slower than that of itinerant electrons [11]. Below we explain the physical meaning of each term.

\mathcal{L}_B represents the single-spin Berry phase, resulting from the curved nature of the SU(2) spin manifold. This term is indispensable for reproducing the skyrmion dynamics that has been obtained in the Landau-Lifshitz-Gilbert equation approach [11].

\mathcal{L}_ψ describes the dynamics of itinerant electrons, where ψ_σ and z_α represent the fermionic field for itinerant electrons and the CP¹ spinon field for localized spins, respectively. As mentioned above, spins of itinerant electrons follow those of localized electrons in the strong Hund's-rule coupling limit. This constraint provides dynamics of itinerant electrons with an effective internal flux, originating from the curvature in the spin space. Namely, their orbital motion is affected by an effective Aharonov-Bohm phase or the Berry-phase connection, described by an internal U(1) gauge field a_μ with $\mu = \tau, x, y$. Physically, this gauge field represents spin chirality fluctuations, and couples to electrons with opposite signs of coupling constants for "spin" up and down. In contrast, A_μ represents the external electromagnetic potential which linearly couples to electric charge/current density of itinerant electrons. μ_τ is the chemical potential, which is determined to fix the total number of itinerant electrons.

t is the hopping energy of itinerant electrons. The last term represents a spin vector potential, which originates from the Rashba spin-orbit coupling λ_{so} . This provides an interaction between the spin and the "spin current", and thus quenches the spin direction to that of the momentum or spin current.

\mathcal{L}_z describes the dynamics of the CP¹ spinon field z_α for localized spins, in other words, their directional (angular) fluctuations. In ferromagnets, this spinon dynamics produces the magnon excitations, which exhibits the $\omega \propto k^2$ dispersion relation in the SU(2) symmetric case. We have introduced the spin density of itinerant electrons, $\rho_s = \langle \sum_\sigma \sigma \psi_\sigma^\dagger \psi_\sigma \rangle$. We have an additional term generated by the Rashba spin-orbit coupling λ_{so} .

Note that the skyrmion configuration creates a non-trivial background potential for the Berry gauge connection a_μ and thus a fictitious internal magnetic field in the z direction. This affects the dynamics/transport of itinerant electrons. Indeed, even if the skyrmion is static, it produces the anomalous Hall current of itinerant electrons [21, 22]. Actually, this is one side of the mutual feedback effect between the skyrmion and the fermionic matter. On the other side, the topologically induced anomalous Hall current is accompanied by the dissipationless skyrmion Hall current, when the skyrmions are depinned intrinsic objects, as we will show later.

The skyrmion motion can be uncovered from an effective action for both U(1) Berry gauge fields a_μ and electromagnetic fields A_μ , integrating over electrons ψ_σ and spinons z_σ in the skyrmion background. We separate the Berry gauge field into two pieces which correspond to

its classical configuration and quantum-fluctuation part, respectively. The classical configuration of the Berry gauge field is determined from an equation of motion for spinons, where dynamics of spinons is taken into account classically. The skyrmion solution of the spinon field gives rise to an effective magnetic field for electrons, given by the following relation of $a_\mu^c = -\frac{i}{2}[z_\sigma^{c\dagger}(\partial_\mu z_\sigma^c) - (\partial_\mu z_\sigma^{c\dagger})z_\sigma^c]$, where the superscript c denotes “classical”. Inserting the effective magnetic field into the Schrodinger

equation for electrons, we construct the space of wave functions, well fitted to the skyrmion potential. Then, we can integrate over electrons, and expand the resulting logarithmic term up to the second order for gauge fluctuations. The whole procedure is shown in section IV. This gives rise to not only the Maxwell Lagrangian \mathcal{L}_M but also the spatially dependent Chern-Simons action \mathcal{L}_{CS} for the gauge-field dynamics,

$$\begin{aligned}\mathcal{L}_{eff} &= \mathcal{L}_B + \mathcal{L}_M + \mathcal{L}_{CS}, \\ \mathcal{L}_M &= \frac{1}{2} \begin{pmatrix} \delta a_i & A_i \end{pmatrix} \begin{pmatrix} \sigma_{ss}|\partial_\tau| + \chi_{ss}(-\partial^2) & \sigma_{sc}|\partial_\tau| + \chi_{sc}(-\partial^2) \\ \sigma_{sc}|\partial_\tau| + \chi_{sc}(-\partial^2) & \sigma_{cc}|\partial_\tau| + \chi_{cc}(-\partial^2) \end{pmatrix} P_{ij}^T \begin{pmatrix} \delta a_j \\ A_j \end{pmatrix}, \\ \mathcal{L}_{CS} &= i \frac{\Theta_{ss}(\mathbf{x} - \mathbf{X})}{2\pi} \epsilon_{\mu\nu\lambda} \delta a_\mu \partial_\nu \delta a_\lambda + i \frac{\Theta_{sc}(\mathbf{x} - \mathbf{X})}{\pi} \epsilon_{\mu\nu\lambda} \delta a_\mu \partial_\nu A_\lambda + i \frac{\Theta_{cc}(\mathbf{x} - \mathbf{X})}{2\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \end{aligned} \quad (2)$$

where δa_μ represents the quantum-fluctuation part.

In the Maxwell Lagrangian \mathcal{L}_M , σ_{ss} , σ_{cc} , σ_{sc} , and χ_{ss} , χ_{cc} , χ_{sc} are conductivities and diamagnetic susceptibilities, associated with spin-current–spin-current, charge-current–charge-current, and spin-current–charge-current correlation functions, respectively. Electrons are assumed to be in the diffusive regime, resulting in the $z = 2$ dynamics for gauge fluctuations, where z is the dynamical exponent to represent the dispersion relation, $\omega \propto k^z$. P_{ij}^T is the projection operator for transverseness of the gauge dynamics, given by $P_{ij}^T = \delta_{ij} + \partial_i \partial_j / (-\partial^2)$, where $i, j = x, y$ and $-\partial^2 = -\partial_x^2 - \partial_y^2$. Dynamics of the temporal part δa_τ can be neglected in the low energy limit because such fluctuations are gapped and decoupled with spatial fluctuations in the Coulomb gauge.

In the Chern-Simons action \mathcal{L}_{CS} , $\Theta_{ss}(\mathbf{x} - \mathbf{X})$ denotes a local spinon-Hall conductance, given by the transverse spin-current–spin-current correlation function. Namely, it describes the spin Hall current generated by the magnetic field gradient. $\Theta_{cc}(\mathbf{x} - \mathbf{X})$ is a local charge-Hall conductance, given by the transverse charge-current–charge-current correlation function. $\Theta_{sc}(\mathbf{x} - \mathbf{X})$ is a local spin-Hall conductance, given by the transverse spin-current–charge-current correlation function. \mathbf{X} represents the spatial coordinate of the skyrmion core, which should be distinguished from that of the fields, i.e., \mathbf{x} . Actually, the skyrmion contribution to the charge Hall current decays with the distance $|\mathbf{x} - \mathbf{X}|$ from the skyrmion core. In particular, the charge Hall current vanishes at the long distance if the relativistic spin-orbit coupling λ_{so} is absent. All coefficients in this effective field theory will be found in section IV.

Based on Eq. (2), we investigate the skyrmion dynamics under an external dc electric field applied along the x direction. Since we are interested only in the transport properties in the linear response to the applied dc elec-

tric field, it is sufficient to treat the constant velocity of the skyrmion core. To facilitate the calculation on the coupling between itinerant electrons and localized spins, it is convenient to introduce the frame moving with the skyrmion core at the origin. In this skyrmion moving frame, the time derivative and the time component of the gauge field are transformed as

$$\partial_\tau \longrightarrow \partial_\tau - \mathbf{v}_r \cdot \partial_r, \quad \delta a_\tau \longrightarrow \delta a_\tau - \mathbf{v}_r \cdot \delta \mathbf{a}_r, \quad (3)$$

where \mathbf{v}_r is the constant skyrmion velocity driven by the external electric field, which will be determined self-consistently below. Note that we implicitly ignore the modification of the shape of the static single-skyrmion configuration, which requires more careful self-consistent treatment of the coupling between itinerant electrons and localized spins, but does not spoil the topological origin of our skyrmion Hall current totally.

B. Skyrmion dynamics under electric field

Taking the derivative of \mathcal{L}_{eff} with respect to A_μ and δa_μ , we obtain an equation of motion for the U(1) Berry gauge field and that for the electromagnetic field, respectively,

$$\begin{aligned} \chi_s \partial^2 \delta a_i + \chi_{cs} \partial^2 A_i &= 2(S - M)v_i \\ &+ \sigma_{ss} \delta e_i - \frac{\Theta_{ss}}{\pi} \epsilon_{ij} \delta e_j - \frac{\Theta_{sc}}{2\pi} \epsilon_{ij} E_j, \\ \chi_c \partial^2 A_i + \chi_{cs} \partial^2 \delta a_i &= -\rho_{el} v_i \\ &+ \sigma_{cc} E_i - \frac{\Theta_{cc}}{\pi} \epsilon_{ij} E_j - \frac{\Theta_{sc}}{2\pi} \epsilon_{ij} \delta e_j \end{aligned} \quad (4)$$

where the imaginary time has been replaced with the real time. The terms linearly proportional to the skyrmion velocity v_i originate from the Berry-phase term in the

moving frame. $M = \langle \sum_{\sigma} \sigma \psi_{\sigma}^{\dagger} \psi_{\sigma} + \rho_s \sum_{\sigma} z_{\sigma}^{\dagger} z_{\sigma} \rangle$ corresponds to the magnetization density, which effectively reduces the coefficient of the Berry phase term and suppresses that of the skyrmion velocity. $\rho_{el} = \langle \sum_{\sigma} \psi_{\sigma}^{\dagger} \psi_{\sigma} \rangle$ is the charge density. E_j is an external electric field and $\delta e_j = \epsilon_{j\mu\nu} \partial_{\mu} \delta a_{\nu}$ is an internal electric field.

Physics of these Maxwell-Chern-Simons equations can be understood as follows. Recalling the structure of the Maxwell equation, one can construct two constituent equations, which relate “spin” and “charge” currents with both internal and external electric fields,

$$\begin{aligned} \sum_{\sigma} \sigma J_{i\sigma}^{\psi} &= \sigma_{ss} \delta e_i - \frac{\Theta_{ss}}{\pi} \epsilon_{ij} \delta e_j - \frac{\Theta_{sc}}{2\pi} \epsilon_{ij} E_j, \\ \sum_{\sigma} J_{i\sigma}^{\psi} &= \sigma_{cc} E_i - \frac{\Theta_{cc}}{\pi} \epsilon_{ij} E_j - \frac{\Theta_{sc}}{2\pi} \epsilon_{ij} \delta e_j, \end{aligned} \quad (5)$$

where $J_{i\sigma}^{\psi}$ represents the current of itinerant electrons with the spin index σ flowing in the j -direction, given by

$$\begin{aligned} J_{j\sigma}^{\psi} &= -it[\psi_{\sigma}^{\dagger}\{(\partial_j - i\sigma(\lambda_{so}/t)z_{\alpha}^{\dagger}\sigma_{\alpha\beta}^j z_{\beta})\psi_{\sigma}\} \\ &\quad - \{(\partial_j + i\sigma(\lambda_{so}/t)z_{\alpha}^{\dagger}\sigma_{\alpha\beta}^j z_{\beta})\psi_{\sigma}^{\dagger}\}\psi_{\sigma}]. \end{aligned} \quad (6)$$

The spin conductivity σ_{ss} vanishes in the paramagnetic phase without the spin-orbit coupling, while in the ferromagnetic phase with the Zeeman splitting $J_H S$, it is finite and the spin current of itinerant electrons is generated by the external electric field E_i . Equation (5) generalizes the standard constituent relation in metals, where dynamics of conduction electrons are in the diffusive regime, introducing the Chern-Simons contribution into the equation, which plays an essential role for the mutual feedback effect between skyrmions and itinerant electrons. The presence of the Chern-Simons term in the constituent equation confirms the anomalous Hall effect

of itinerant electrons in the metallic ferromagnet, as discussed before.

It is straightforward to solve coupled equations (4) when the spatial dependence for both gauge fields is neglected. Performing integration of $\int dx \int dy$ in Eq. (4), we obtain

$$\begin{aligned} -2(S - M)v_i &= \sigma_{ss} \delta e_i - \frac{\sigma_{ss}^H}{\pi L^2} \epsilon_{ij} \delta e_j - \frac{\sigma_{sc}^H}{2\pi L^2} \epsilon_{ij} E_j, \\ \rho_{el} v_i &= \sigma_{cc} E_i - \frac{\sigma_{cc}^H}{\pi L^2} \epsilon_{ij} E_j - \frac{\sigma_{sc}^H}{2\pi L^2} \epsilon_{ij} \delta e_j, \end{aligned} \quad (7)$$

where

$$\begin{aligned} \sigma_{ss}^H &= \int dx \int dy \Theta_{ss}(\mathbf{x}), \\ \sigma_{cc}^H &= \int dx \int dy \Theta_{cc}(\mathbf{x}), \\ \sigma_{sc}^H &= \int dx \int dy \Theta_{sc}(\mathbf{x}) \end{aligned} \quad (8)$$

are Hall conductivities with L being the linear spatial dimension of the system.

Coupled equations (7) describe both the internal electric field δe_i and the skyrmion velocity v_i as a function of the external electric field E_i . We find the following expression for the internal electric field

$$\begin{aligned} \begin{pmatrix} e_x \\ e_y \end{pmatrix} &= \left(\frac{\sigma_{sc}^H}{2\pi L^2} \right)^{-2} \left\{ \rho_{el} \begin{pmatrix} 0 & \frac{\sigma_{sc}^H}{2\pi L^2} \\ -\frac{\sigma_{sc}^H}{2\pi L^2} & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} -\frac{\sigma_{sc}^H}{2\pi L^2} \frac{\sigma_{cc}^H}{\pi L^2} & -\frac{\sigma_{sc}^H}{2\pi L^2} \sigma_{cc} \\ \frac{\sigma_{sc}^H}{2\pi L^2} \sigma_{cc} & -\frac{\sigma_{sc}^H}{2\pi L^2} \frac{\sigma_{cc}^H}{\pi L^2} \end{pmatrix} \begin{pmatrix} E_x \\ 0 \end{pmatrix} \right\}. \end{aligned} \quad (9)$$

Inserting this expression into Eq. (7), we obtain the skyrmion velocity as a function of the external electric field E_x ,

$$\begin{aligned} v_x &= \frac{\left((S - M)\sigma_{sc}^H + \rho_{el}\sigma_{ss}^H \right) \left(\sigma_{ss}\sigma_{cc}^H + \sigma_{cc}\sigma_{ss}^H \right) + \rho_{el}\sigma_{ss} \left(\pi^2 L^4 \sigma_{ss}\sigma_{cc} - \sigma_{ss}^H \sigma_{cc}^H + \frac{\sigma_{sc}^{H2}}{4} \right)}{\left((S - M)\sigma_{sc}^H + \rho_{el}\sigma_{ss}^H \right)^2 + \pi^2 L^4 (\rho_{el}\sigma_{ss})^2} E_x, \\ v_y &= \frac{-\frac{1}{\pi L^2} \left((S - M)\sigma_{sc}^H + \rho_{el}\sigma_{ss}^H \right) \left(\pi^2 L^4 \sigma_{ss}\sigma_{cc} - \sigma_{ss}^H \sigma_{cc}^H + \frac{\sigma_{sc}^{H2}}{4} \right) + \rho_{el}\sigma_{ss} \left(\pi^2 L^4 \sigma_{ss}\sigma_{cc} - \sigma_{ss}^H \sigma_{cc}^H \right)}{\left((S - M)\sigma_{sc}^H + \rho_{el}\sigma_{ss}^H \right)^2 + \pi^2 L^4 (\rho_{el}\sigma_{ss})^2} E_x. \end{aligned} \quad (10)$$

To understand the above expression, we consider two limiting cases. First, we take the limit of $\sigma_{ss}^H = \sigma_{cc}^H = \sigma_{sc}^H = 0$, resulting in

$$v_x = \frac{\sigma_{cc}}{\rho_{el}} E_x, \quad v_y = \frac{\sigma_{cc}}{\rho_{el}} E_x. \quad (11)$$

The skyrmion current is driven to not only the same direction as the applied electric field but also the orthogo-

nal direction corresponding to the Hall motion.

Second, we take another limit of $\sigma_{ss} = \sigma_{cc} = \sigma_{sc} = 0$, corresponding to an insulator. Then, we find

$$\begin{aligned} v_x &= 0, \\ v_y &= \frac{1}{\pi L^2} \frac{\sigma_{ss}^H \sigma_{cc}^H - \frac{\sigma_{sc}^{H2}}{4}}{(S - M)\sigma_{sc}^H + \rho_{el}\sigma_{ss}^H} E_x. \end{aligned} \quad (12)$$

This is a remarkable result. Although the skyrmion Hall current induced by the electric field vanishes in the thermodynamic limit, the nature of the skyrmion Hall current is dissipationless. We cannot find any coefficients associated with conductivity, giving rise to dissipation. This certainly originates from the Chern-Simons terms, an essential feature of the “self-consistent” treatment.

It is interesting to observe that the dissipationless skyrmion Hall current may not vanish in the thermodynamic limit if the spin-orbit interaction is introduced. The Hall coefficient without the spin-orbit coupling is proportional to the density of skyrmions, thus $\sim L^{-2}$ in the case of the single skyrmion. As a result, Hall conductivities of σ_{ss}^H , σ_{cc}^H , and σ_{sc}^H in Eq. (12) are constants even after the spatial integration. However, the spin-orbit interaction can give rise to a finite value for the Chern-Simons coefficients in the thermodynamic limit, thus its integral value corresponding to σ_{ss}^H , σ_{cc}^H , and σ_{sc}^H will be proportional to L^2 . Then, we obtain the dissipationless skyrmion Hall current in ferromagnetic insulators with the spin-orbit coupling.

C. Discussion : Dissipationless skyrmion Hall current in the surface state of three dimensional topological insulators

We discuss the dissipationless skyrmion Hall current in the surface state of three dimensional topological insulators, where magnetic impurities are deposited. An effective field theory for surface Dirac electrons with localized spins is given by

$$\mathcal{S} = \mathcal{S}_B + \int d^3\mathbf{x} \left\{ \bar{\psi} \left(i\hat{\mathbf{D}} - m\vec{\mathbf{n}} \cdot \vec{\boldsymbol{\tau}} \right) \psi + \frac{1}{2e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right\}. \quad (13)$$

\mathcal{S}_B is the single-spin Berry phase term to appear from the coherent-state representation for the localized spin $\vec{\mathbf{n}}$ in the path-integral quantization. ψ represents the surface Dirac fermion in the irreducible representation, and the covariant derivative is $i\hat{\mathbf{D}} \equiv \gamma_\mu (i\partial_\mu + A_\mu)$, where γ_μ with $\mu = \tau, x, y$ is two by two Dirac matrices and A_μ is an external electromagnetic vector potential. $m > 0$ is an effective coupling constant between surface Dirac fermions and deposited magnetic impurities. Compared with the double exchange model, the only difference is that non-relativistic electrons are replaced with Dirac fermions.

Integrating over Dirac electrons and performing the gradient expansion for the resulting logarithmic term [23], one finds the following effective action

$$\begin{aligned} \mathcal{S}_{eff} = & \mathcal{S}_B + \int d^3\mathbf{x} \left(\frac{m}{8\pi} (\partial_\mu \vec{\mathbf{n}})^2 + iA_\mu J_\mu \right. \\ & \left. + \frac{i}{4\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{1}{2e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right) + i\pi\vartheta[\vec{\mathbf{n}}], \end{aligned} \quad (14)$$

where dynamics of localized spins is governed by the nonlinear σ model with additional terms. The conserved

current

$$J_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\lambda} \vec{\mathbf{n}} \cdot \partial_\nu \vec{\mathbf{n}} \times \partial_\lambda \vec{\mathbf{n}}$$

corresponds to the skyrmion current, discussed before. The minimal coupling between the skyrmion current and the electromagnetic field implies that a skyrmion carries an electric charge, where a normalizable fermion zero mode exists on the topological soliton, inducing a fermionic charge. On the other hand, the last term shows the geometric phase, identified with the Hopf term, which determines the statistics of the topological soliton and its spin quantum number. These two terms are well known in the field theory, referred as the quantum anomaly, where the induced fermionic charge for a soliton is a phenomenon due to the local anomaly while the topological phase is due to the global anomaly [23, 24]. Mathematically speaking, the existence of such topological terms is guaranteed by the fundamental property of the elliptic operator, the Dirac operator in the present case, called the Atiya-Singer index theorem [25].

One can express the above effective field theory as follows, resorting to the CP^1 representation,

$$\begin{aligned} \mathcal{S}_{eff} = & \mathcal{S}_B + \int d^3\mathbf{x} \left(\frac{1}{2g^2} |(\partial_\mu - ia_\mu)z_\sigma|^2 + \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right. \\ & \left. + \frac{i}{4\pi} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{i}{4\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{1}{2e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right) \end{aligned} \quad (15)$$

with $g^2 \propto 1/m$, where the skyrmion current is $J_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda$ and the topological phase is $\vartheta[\vec{\mathbf{n}}] = \frac{1}{4\pi^2} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$. This effective field theory for dynamics of localized spins on the topological surface is essentially the same as our previous effective field theory for dynamics of localized spins in the presence of non-relativistic electrons except for the fact that coefficients in topological terms, i.e., Chern-Simons terms have finite values in the thermodynamic limit. As a result, we find the dissipationless skyrmion Hall current in the surface state of three dimensional topological insulators, although surface Dirac electrons become gapped and insulating due to time reversal symmetry breaking. Furthermore, the electric charge of the skyrmion will give rise to an additional contribution for the Hall voltage beyond that from gapped Dirac fermions.

It seems to be clear that the dissipationless skyrmion Hall current will survive on the topological surface state in the thermodynamic limit. However, it is difficult to guarantee such a phenomenon in the case of non-relativistic electrons with the spin-orbit interaction. First of all, it should be noted that our effective field theory [Eq. (1)] is valid only in the limit of $J_H \gg t > \lambda_{so}$. In the $J_H \rightarrow 0$ limit $\rho_s = \left\langle \sum_\sigma \sigma \psi_\sigma^\dagger \psi_\sigma \right\rangle \propto J_H S$ will vanish. As a result, we lose terms to describe dynamics of local spins in the effective Lagrangian. This results from the $U(1)$ projection [Eq. (A6)] of the original $SU(2)$ effective theory [Eq. (A5)]. On the other hand, the $J_H \rightarrow 0$

limit recovers the Rashba model via an appropriate gauge transformation in Eq. (A5). In the strong Hund-coupling limit the spin-orbit interaction gives rise to an additional internal magnetic field for itinerant electrons, as shown in Eq. (1). Inserting the skyrmion configuration into the spin-orbit induced gauge field, we can see that the internal effective magnetic flux given by the spin-orbit interaction, $\frac{1}{L^2} \int d^2\mathbf{r} [\partial_x(z_\alpha^\dagger \sigma_{\alpha\beta}^y z_\beta) - \partial_y(z_\alpha^\dagger \sigma_{\alpha\beta}^x z_\beta)]$, decreases as $1/L^2$, where L is the system size. The effective magnetic flux via the gauge field of the skyrmion configuration, $\frac{1}{L^2} \int d^2\mathbf{r} (\partial_x a_y - \partial_y a_x)$, is also proportional to $1/L^2$ as discussed before. As a result, the spin-orbit interaction does not allow the dissipationless skyrmion Hall current to survive in the thermodynamic limit when J_H is assumed to be large in Eq. (1).

In the $J_H \rightarrow 0$ limit the situation is more tricky. As well known, the Rashba model ($J_H = 0$ in Eq. (A1)) shows the spin Hall effect [26], where disorder effects are not taken into account. When ferromagnetic interactions are turned on, a helical ordered state is expected to appear. The question is whether the spin Hall effect survives or not in the helical ordered state. When the spin Hall effect exists, the dissipationless skyrmion Hall current will be observed in the thermodynamic limit. Unfortunately, we do not have any definite answer because the helical order will change the electron dispersion of the Rashba model, which can spoil the spin Hall effect. In particular, the ordering wave vector may be incommensurate generically, making the problem much complicated. We leave this interesting problem as a future work.

III. COMPARISON WITH THE LANDAU-LIFSHITZ-GILBERT EQUATION APPROACH

The well known Landau-Lifshitz-Gilbert equation

$$\frac{\partial \vec{S}}{\partial t} = \gamma \vec{B} \times \vec{S} - \frac{\alpha}{S} \vec{S} \times \frac{\partial \vec{S}}{\partial t} \quad (16)$$

is generalized as

$$\begin{aligned} \frac{\partial \vec{S}}{\partial t} = & \gamma \vec{B} \times \vec{S} - \frac{\alpha}{S} \vec{S} \times \frac{\partial \vec{S}}{\partial t} \\ & - \frac{a^3}{2eS} (\vec{j}_s \cdot \vec{\nabla}) \vec{S} - \frac{a^3 \beta}{2eS^2} [\vec{S} \times (\vec{j}_s \cdot \vec{\nabla}) \vec{S}] \end{aligned} \quad (17)$$

in the presence of spin current [11]. The first term in Eq. (16) is the standard precession term with $\gamma = g\mu_B/\hbar > 0$, where $g = 2$ is the g -factor and μ_B is the Bohr magneton. \vec{B} is an effective magnetic field. The second term in Eq. (16) is the Gilbert damping term, phenomenologically introduced, where α is a damping coefficient and S is spin. The third term in Eq. (17) describes the spin-transfer torque from the spin current, where \vec{j}_s is the spin current and a is the Bohr radius. The fourth term in Eq. (17) represents another torque contribution, perpendicular to the spin-transfer torque, argued to result

from spin relaxation of conduction electrons. This term is called the β term due to the coefficient β .

One can derive an equation of motion for domain walls, vortices, and skyrmions from this generalized Landau-Lifshitz-Gilbert equation, resorting to the collective coordinate method [12]. Suppose that the right hand side of Eq. (16) or Eq. (17) can be derived from an effective Hamiltonian H_{eff} . Then, one can write down an equation of motion for spin dynamics as follows

$$\frac{\partial \vec{S}}{\partial t} = -\vec{S} \times \frac{\delta H_{eff}}{\delta \vec{S}}.$$

Multiplying $\vec{S} \times \frac{\partial \vec{S}}{\partial \xi_i}$ to both sides, we obtain

$$\vec{S} \cdot \frac{\partial \vec{S}}{\partial \xi_i} \times \frac{\partial \vec{S}}{\partial t} = -\frac{\partial \vec{S}}{\partial \xi_i} \cdot \frac{\delta H_{eff}}{\delta \vec{S}},$$

where the orthogonality condition $\vec{S} \cdot \frac{\partial \vec{S}}{\partial \xi_i} = 0$ is used. ξ_i is called the collective coordinate, representing the core position of the skyrmion (domain wall or vortex), where $i = x, y$. Integrating both sides over the two dimensional space area \mathbf{A} , we reach the following expression for the skyrmion dynamics

$$4\pi Q \sum_{j=x,y} \epsilon_{ij} \frac{\partial \xi_j}{\partial t} = - \int_{\mathbf{A}} d^2\mathbf{r} \frac{\delta H_{eff}}{\delta \xi_i}, \quad (18)$$

where Q is an integer identified with the skyrmion number

$$4\pi Q \epsilon_{ij} = \int_{\mathbf{A}} d^2\mathbf{r} \vec{S} \cdot \frac{\partial \vec{S}}{\partial \xi_i} \times \frac{\partial \vec{S}}{\partial \xi_j}. \quad (19)$$

The skyrmion or vortex dynamics has been studied in this framework, starting from essentially the same effective action as the present model Hamiltonian [Eq. (A1)]

$$\begin{aligned} S_{eff} = & \int d^3x \left\{ c_\sigma^\dagger \left(-i\partial_t - \frac{\partial_{\mathbf{r}}^2}{2m} \right) c_\sigma - J_H \vec{n} \cdot c_\alpha^\dagger \vec{\sigma}_{\alpha\beta} c_\beta \right. \\ & \left. + S\dot{\phi}(1 - \cos\theta) + \frac{JS^2}{2} (\partial_{\mathbf{r}} \vec{n})^2 + V(\vec{n}) \right\}, \end{aligned}$$

where $\vec{n} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$ represents a spin direction and $V(\vec{n})$ introduces spin anisotropy. J is the ferromagnetic exchange coupling constant and $\dot{\phi} = \partial_t \phi$.

Performing the same unitary transformation as Eq. (A3), one obtains the same effective action as Eq. (A5) except for the Rashba spin-orbit coupling term

$$\begin{aligned} S_{eff} = & \int d^3x \left\{ \psi_\sigma^\dagger \left(-i(\partial_t - ia_t) - \frac{(\partial_{\mathbf{r}} - ia_{\mathbf{r}})^2}{2m} \right) \psi_\sigma \right. \\ & \left. - J_H \sigma \psi_\sigma^\dagger \psi_\sigma + S\dot{\phi}(1 - \cos\theta) + \frac{JS^2}{2} (\partial_{\mathbf{r}} \vec{n})^2 + V(\vec{n}) \right\}, \end{aligned}$$

where the Berry gauge field is given by $a_{\mathbf{r}} = \vec{a}_{\mathbf{r}} \cdot \vec{\sigma} = -i\mathbf{U}^\dagger \partial_{\mathbf{r}} \mathbf{U}$. As intensively discussed before, an essential effect is given by the spin chirality fluctuation (Berry gauge field) to the spin-current minimal coupling term

$$H_{ST} = \int d^3x \vec{j}_s \cdot \frac{\vec{\nabla} \phi}{2} (1 - \cos\theta),$$

where the gauge field is represented with two angles.

Based on this effective action, one can derive an equation of motion for the vortex or skyrmion dynamics [27]

$$\vec{G} \times (\vec{v}_s - \partial_t \vec{\xi}) = -\frac{\partial U(\vec{\xi})}{\partial \vec{\xi}} - \alpha \partial_t \vec{\xi}. \quad (20)$$

$\vec{G} = \hat{e}_z S \int d^3x \vec{n} \cdot (\partial_x \vec{n} \times \partial_y \vec{n}) = 4\pi S Q \hat{e}_z$ expresses the internal magnetic flux given by the skyrmion charge Q . \vec{v}_s represents the velocity associated with the spin current of itinerant electrons. $U(\vec{\xi})$ can be regarded as a pinning potential, and the first term in the right hand side describes the associated pinning force. α is the Gilbert damping constant.

Comparing this equation of motion with Eq. (17), the extended Landau-Lifshitz-Gilbert equation, we see that $\gamma \vec{B} \times \vec{S}$ and $-\frac{a^3 \beta}{2eS^2} [\vec{S} \times (\vec{J}_s \cdot \vec{\nabla}) \vec{S}]$ in Eq. (17) are not introduced in this equation of motion. $\vec{G} \times \partial_t \vec{\xi}$ is associated with $\frac{\partial \vec{S}}{\partial t}$, and $\vec{G} \times \vec{v}_s$ corresponds to $\frac{a^3}{2eS} (\vec{J}_s \cdot \vec{\nabla}) \vec{S}$. $\alpha \partial_t \vec{\xi}$ results from $\frac{\alpha}{S} \vec{S} \times \frac{\partial \vec{S}}{\partial t}$.

Each term in our Maxwell-Chern-Simons approach [Eq. (4)] has its partner in Eq. (20) except for Chern-Simons terms proportional to Θ_{cc} and Θ_{sc} . Considering the following constituent equations

$$\begin{aligned} \sum_{\sigma} \sigma J_{i\sigma}^{\psi} &= \sigma_{ss} \delta e_i - \frac{\Theta_{ss}}{\pi} \epsilon_{ij} \delta e_j, \\ \sum_{\sigma} J_{i\sigma}^{\psi} &= \sigma_{cc} E_i, \end{aligned}$$

we obtain

$$\begin{aligned} v_x &= \frac{\rho_{el} \sigma_{ss}^{H2} \sigma_{cc} + \pi^2 L^4 \rho_{el} \sigma_{ss}^2 \sigma_{cc}}{\rho_{el}^2 \sigma_{ss}^{H2} + \pi^2 L^4 \rho_{el}^2 \sigma_{ss}^2} E_x, \\ v_y &= \frac{-\pi L^2 \rho_{el} \sigma_{ss}^H \sigma_{ss} \sigma_{cc} + \pi^2 L^4 \rho_{el} \sigma_{ss}^2 \sigma_{cc}}{\rho_{el}^2 \sigma_{ss}^{H2} + \pi^2 L^4 \rho_{el}^2 \sigma_{ss}^2} E_x \end{aligned}$$

from our Maxwell-Chern-Simons approach. On the other hand, Eq. (20) gives rise to the following skyrmion motion

$$\begin{aligned} v_x &= \frac{(4\pi S Q)^2 (\sigma_{cc} / \rho_{el})}{(4\pi S Q)^2 + \alpha^2} E_x, \\ v_y &= -\frac{(4\pi S Q) \alpha (\sigma_{cc} / \rho_{el})}{(4\pi S Q)^2 + \alpha^2} E_x, \end{aligned}$$

where the drift velocity of the spin current is given by $\vec{v}_s = (\sigma_{cc} / \rho_{el}) E_x \hat{x}$. We note that the charge current in terms of the ψ_{σ} field contains the contribution of the spin current in terms of the electron field c_{σ} . In this respect the charge conductivity σ_{cc} differs from the actual electrical conductivity. Identifying $4\pi S Q$ with $\rho_{el} \sigma_{ss}^H / L^2$ and α with $\pi \rho_{el} \sigma_{ss}$, the above expression becomes

$$\begin{aligned} v_x &= \frac{\rho_{el}^2 \sigma_{ss}^{H2} \sigma_{cc}}{\rho_{el}^2 \sigma_{ss}^{H2} + \pi^2 L^4 \rho_{el}^2 \sigma_{ss}^2} E_x, \\ v_y &= -\frac{\pi L^2 \rho_{el} \sigma_{ss}^H \sigma_{ss} \sigma_{cc}}{\rho_{el}^2 \sigma_{ss}^{H2} + \pi^2 L^4 \rho_{el}^2 \sigma_{ss}^2} E_x. \end{aligned}$$

This result coincides with the first two contributions in the Maxwell-Chern-Simons-equation approach.

IV. EVALUATION OF CHERN-SIMONS TERMS

A. Current-current correlation functions

An important task is to show that the Chern-Simons coefficients are non-vanishing. In this section we evaluate all current-current correlation functions explicitly. Integrating over itinerant electrons in \mathcal{L}_{ψ} of Eq. (1), we find the local Chern-Simons action (\mathcal{L}_{CS}) in Eq. (2). Generally, we consider an effective action for gauge fluctuations

$$\mathcal{S}_{eff}^{gauge} = \int d\mathbf{x} \int d\mathbf{x}' \frac{1}{2} \begin{pmatrix} a_{\mu}(\mathbf{x}) & A_{\mu}(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \Pi_{\mu\nu}^{ss}(\mathbf{x}, \mathbf{x}') & \Pi_{\mu\nu}^{sc}(\mathbf{x}, \mathbf{x}') \\ \Pi_{\mu\nu}^{cs}(\mathbf{x}, \mathbf{x}') & \Pi_{\mu\nu}^{cc}(\mathbf{x}, \mathbf{x}') \end{pmatrix} \begin{pmatrix} a_{\nu}(\mathbf{x}') \\ A_{\nu}(\mathbf{x}') \end{pmatrix}, \quad (21)$$

where the gauge kernel matrix is given by

$$\begin{aligned} \Pi_{\mu\nu}^{ss}(\mathbf{x}, \mathbf{x}') &= t^2 \left\langle T_{\tau} \left\{ [\sigma J_{\sigma\mu}(\mathbf{x})] [\sigma' J_{\sigma'\nu}(\mathbf{x}')] \right\} \right\rangle_c - 4it\lambda_{so} \left\langle T_{\tau} \left\{ [n_{\mu}^c(\mathbf{x}) \rho_{\sigma}(\mathbf{x})] [\sigma' J_{\sigma'\nu}(\mathbf{x}')] \right\} \right\rangle_c - \left(\mathbf{x} \leftrightarrow \mathbf{x}', \mu \leftrightarrow \nu \right) \\ &\quad - 16\lambda_{so}^2 \left\langle T_{\tau} \left\{ [n_{\mu}^c(\mathbf{x}) \rho_{\sigma}(\mathbf{x})] [n_{\nu}^c(\mathbf{x}') \rho_{\sigma'}(\mathbf{x}')] \right\} \right\rangle_c, \\ \Pi_{\mu\nu}^{cc}(\mathbf{x}, \mathbf{x}') &= t^2 \left\langle T_{\tau} \left\{ [J_{\sigma\mu}(\mathbf{x})] [J_{\sigma'\nu}(\mathbf{x}')] \right\} \right\rangle_c - 4it\lambda_{so} \left\langle T_{\tau} \left\{ [n_{\mu}^c(\mathbf{x}) \sigma \rho_{\sigma}(\mathbf{x})] [J_{\sigma'\nu}(\mathbf{x}')] \right\} \right\rangle_c - \left(\mathbf{x} \leftrightarrow \mathbf{x}', \mu \leftrightarrow \nu \right) \\ &\quad - 16\lambda_{so}^2 \left\langle T_{\tau} \left\{ [n_{\mu}^c(\mathbf{x}) \sigma \rho_{\sigma}(\mathbf{x})] [n_{\nu}^c(\mathbf{x}') \sigma' \rho_{\sigma'}(\mathbf{x}')] \right\} \right\rangle_c, \\ \Pi_{\mu\nu}^{sc}(\mathbf{x}, \mathbf{x}') &= t^2 \left\langle T_{\tau} \left\{ [\sigma J_{\sigma\mu}(\mathbf{x})] [J_{\sigma'\nu}(\mathbf{x}')] \right\} \right\rangle_c - 4it\lambda_{so} \left\langle T_{\tau} \left\{ [n_{\mu}^c(\mathbf{x}) \rho_{\sigma}(\mathbf{x})] [J_{\sigma'\nu}(\mathbf{x}')] \right\} \right\rangle_c - \left(\mathbf{x} \leftrightarrow \mathbf{x}', \mu \leftrightarrow \nu \right) \\ &\quad - 16\lambda_{so}^2 \left\langle T_{\tau} \left\{ [n_{\mu}^c(\mathbf{x}) \rho_{\sigma}(\mathbf{x})] [n_{\nu}^c(\mathbf{x}') \sigma' \rho_{\sigma'}(\mathbf{x}')] \right\} \right\rangle_c, \end{aligned} \quad (22)$$

where the summation for $\sigma = \pm$ should be performed. $n_{\mu}^c = \frac{1}{2} z_{\alpha}^{\dagger} \sigma_{\alpha\beta}^{\mu} z_{\beta}^c$ represents the skyrmion configuration

of the spin direction μ , and

$$\begin{aligned} J_{\sigma\mu}(\mathbf{x}) &= \psi_{\sigma}^{\dagger}(\mathbf{x}) [\partial_{\mu} \psi_{\sigma}(\mathbf{x})] - [\partial_{\mu} \psi_{\sigma}^{\dagger}(\mathbf{x})] \psi_{\sigma}(\mathbf{x}), \\ \rho_{\sigma}(\mathbf{x}) &= \psi_{\sigma}^{\dagger}(\mathbf{x}) \psi_{\sigma}(\mathbf{x}) \end{aligned} \quad (23)$$

denote the current and density with spin σ . $\Pi_{\mu\nu}^{ss}(\mathbf{x}, \mathbf{x}')$, $\Pi_{\mu\nu}^{cc}(\mathbf{x}, \mathbf{x}')$, and $\Pi_{\mu\nu}^{sc}(\mathbf{x}, \mathbf{x}')$ are the spin-current-spin-current, charge-current-charge-current, and spin-current-charge-current correlation functions, respec-

tively. The subscript c represents “connected.”

Applying the Wick’s theorem to the above expression, we obtain

$$\begin{aligned}
\Pi_{\mu\nu}^{ss}(\mathbf{x}, \mathbf{x}') &= -t^2 \sum_{\sigma=\pm} \left[\left\{ \partial_{x_\mu} G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \right\} \left\{ \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right\} - \left\{ \partial_{x_\mu} \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \right\} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right. \\
&\quad - G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \left\{ \partial_{x_\mu} \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right\} + \left\{ \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \right\} \left\{ \partial_{x_\mu} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right\} \Big] \\
&\quad + 4it\lambda_{so}n_\mu^c(\mathbf{x}) \sum_{\sigma=\pm} \sigma \left[G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \left\{ \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right\} - \left\{ \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \right\} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right] + (\mathbf{x} \leftrightarrow \mathbf{x}', \mu \leftrightarrow \nu) \\
&\quad + 16\lambda_{so}^2 n_\mu^c(\mathbf{x}) n_\nu^c(\mathbf{x}') \sum_{\sigma=\pm} G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}), \\
\Pi_{\mu\nu}^{cc}(\mathbf{x}, \mathbf{x}') &= \Pi_{\mu\nu}^{ss}(\mathbf{x}, \mathbf{x}'), \\
\Pi_{\mu\nu}^{sc}(\mathbf{x}, \mathbf{x}') &= -t^2 \sum_{\sigma=\pm} \sigma \left[\left\{ \partial_{x_\mu} G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \right\} \left\{ \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right\} + \left\{ \partial_{x_\mu} \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \right\} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right. \\
&\quad + G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \left\{ \partial_{x_\mu} \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right\} - \left\{ \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \right\} \left\{ \partial_{x_\mu} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right\} \Big] \\
&\quad + 4it\lambda_{so}n_\mu^c(\mathbf{x}) \sum_{\sigma=\pm} \left[G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \left\{ \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right\} - \left\{ \partial_{x'_\nu} G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') \right\} G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}) \right] + (\mathbf{x} \leftrightarrow \mathbf{x}', \mu \leftrightarrow \nu) \\
&\quad + 16\lambda_{so}^2 n_\mu^c(\mathbf{x}) n_\nu^c(\mathbf{x}') \sum_{\sigma=\pm} \sigma G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') G_{\sigma\sigma}(\mathbf{x}', \mathbf{x}), \tag{24}
\end{aligned}$$

where the single particle Green’s function is given by

$$G_{\sigma\sigma}(\mathbf{x}, \mathbf{x}') = -\left\langle T_\tau \left\{ \psi_\sigma(\mathbf{x}) \psi_\sigma^\dagger(\mathbf{x}') \right\} \right\rangle. \tag{25}$$

B. Single particle Green’s function

It is an essential procedure to find the single particle Green’s function in the presence of the single skyrmion potential, where the plane wave basis cannot be applied

due to translational symmetry breaking. An important feature that we try to catch is time reversal symmetry breaking to allow Hall conductivity, where an internal magnetic flux of the skyrmion gives rise to the Hall motion of itinerant electrons. This effect will be introduced, solving the Schrodinger equation in the single skyrmion potential and finding new eigen states.

We start from the Schrodinger equation with the single skyrmion potential in the polar coordinate

$$\begin{aligned}
&(-i\partial_t - \mu_r - J_H S \sigma) \psi_\sigma(r, \phi, t) \\
&-t \left\{ \left(\frac{\partial}{\partial r} - i\sigma \frac{2\lambda_{so}}{ta} \frac{2\xi r}{\xi^2 + r^2} \right)^2 + \frac{1}{r} \left(\frac{\partial}{\partial r} - i\sigma \frac{2\lambda_{so}}{ta} \frac{2\xi r}{\xi^2 + r^2} \right) + \frac{1}{r^2} \left(\partial_\phi + i\sigma \frac{r^2}{\xi^2 + r^2} \right)^2 \right\} \psi_\sigma(r, \phi, t) = 0, \tag{26}
\end{aligned}$$

where ξ is the skyrmion core size. In appendix B1 we show its derivation.

A standard way to solve this kind of equation is to

expand the wave function in a complete basis

$$\psi_\sigma(r, \phi, t) = \int dE_\sigma e^{-iE_\sigma t} \sum_n a_\sigma(n, E_\sigma) \Psi_n^\sigma(r, \phi; E_\sigma), \tag{27}$$

where E_σ is an energy eigen value and n is a good quantum number. $\Psi_n^\sigma(r, \phi; E_\sigma)$ is an eigen state described

by two conserving quantum numbers, n and E_σ , and $a_\sigma(n, E_\sigma)$ is an associated coefficient in the expansion, identified with an annihilation operator in the second quantization expression. The rotational symmetry of the skyrmion potential leads us to the following decomposition

$$\Psi_n^\sigma(r, \phi; E_\sigma) = \mathcal{C} e^{in\phi} F_\sigma^n(r; E_\sigma), \quad (28)$$

where n is identified with an angular momentum and $F_\sigma^n(r; E_\sigma)$ is the corresponding radial wave function. \mathcal{C} is the normalization constant, determined by

$$\int_0^{2\pi} d\phi \int_0^\infty dr r \sum_n \Psi_n^{\sigma*}(r, \phi; E_\sigma) \Psi_n^\sigma(r, \phi; E_\sigma) = \delta(E_\sigma - E_\sigma'). \quad (29)$$

Unfortunately, we fail to find an analytic expression for the radial wave function in the presence of the spin-orbit interaction. Instead, we could obtain the most general expression for the radial wave function in the absence of the spin-orbit coupling

$$\begin{aligned} F_\sigma^n(r) = & C_1(\xi^2 + r^2)^{\frac{1+\sqrt{2}}{2}} r^n \text{HeunC}\left(0, n, \sqrt{2}, \frac{1+n}{2} - \frac{\xi^2}{4} \mathcal{E}_\sigma, \frac{1-\sigma n}{2} + \frac{\xi^2}{4} \mathcal{E}_\sigma, -\frac{r^2}{\xi^2}\right) \\ & + C_2(\xi^2 + r^2)^{\frac{1+\sqrt{2}}{2}} r^{-n} \text{HeunC}\left(0, -n, \sqrt{2}, \frac{1-n}{2} - \frac{\xi^2}{4} \mathcal{E}_\sigma, \frac{1-\sigma n}{2} + \frac{\xi^2}{4} \mathcal{E}_\sigma, -\frac{r^2}{\xi^2}\right), \end{aligned} \quad (30)$$

where $\text{HeunC}\left(0, n, \sqrt{2}, \frac{1+n}{2} - \frac{\xi^2}{4} \mathcal{E}_\sigma, \frac{1-\sigma n}{2} + \frac{\xi^2}{4} \mathcal{E}_\sigma, -\frac{r^2}{\xi^2}\right)$ is a solution of the Heun's confluent equation, regarded as the generalization of the hypergeometric function when an ordinary differential equation contains four singularities [28]. $\mathcal{E}_\sigma \equiv t^{-1}(E_\sigma + \mu_r + J_H S \sigma)$ is an effective energy level. In appendix B2 we discuss the Heun's confluent equation in detail.

The main feature of this wave function lies in the time

reversal symmetry breaking. An effective magnetic flux due to the skyrmion gives rise to an effective Lorentz force for itinerant electrons, resulting in chirality to the electron state. One can see this effect from the difference between two time reversal states with opposite angular momenta. In the asymptotic limit of $r \rightarrow \infty$ the Heun's confluent function becomes

$$F_\sigma^n(r) = C'_1(\xi^2 + r^2)^{\frac{1+\sqrt{2}}{2}} r^n \exp\left(-\frac{\mathcal{E}_\sigma}{4(n+2+\sqrt{2})} r^2\right) + C'_2(\xi^2 + r^2)^{\frac{1+\sqrt{2}}{2}} r^{-n} \exp\left(-\frac{\mathcal{E}_\sigma}{4(-n+2+\sqrt{2})} r^2\right),$$

as shown in appendix B2. This asymptotic wave function reveals that quantum states with negative angular momenta are suppressed due to the effective Lorentz force, imposing the condition that the wave function should not diverge in the $r \rightarrow \infty$ limit. If we consider positive angular momenta in the above expression, the second term allows only $n = 1, 2$ due to the regularity condition, thus forcing C'_2 to vanish identically. On the other hand, the first term allows only $n = -1, -2$ among negative an-

gular momenta, giving rise to $C'_1 = 0$. Considering that either case reaches the same expression, the consistent solution is given by only the first term with positive angular momenta.

It is straightforward to find the single particle Green's function in the new basis. Inserting Eq. (27) with Eq. (28) into the Green's function [Eq. (25)] and performing the Fourier transformation, we obtain

$$G_{\sigma\sigma}(r, r', \phi - \phi', i\omega) = |\mathcal{C}|^2 \int d\mathcal{E}_\sigma \sum_n e^{in(\phi - \phi')} \frac{F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma)}{i\omega - \Sigma_\sigma(i\omega, \mathcal{E}_\sigma) - \mathcal{E}_\sigma}, \quad (31)$$

where the self-energy correction

$$\Sigma_\sigma(i\omega, \mathcal{E}_\sigma) = \Sigma_\sigma^{imp}(i\omega, \mathcal{E}_\sigma) + \Sigma_\sigma^{ele}(i\omega, \mathcal{E}_\sigma)$$

is introduced, resulting from both elastic impurity scattering and inelastic interaction effects with gauge fluctuations, respectively. A way to understand this expression is shown in appendix B3.

C. Longitudinal conductivity

Inserting the Green's function [Eq. (31)] into the conductivity tensor [Eq. (24)] with the following representation

tation

$$\partial_x = \cos \phi \partial_r - \frac{\sin \phi}{r} \partial_\phi, \quad \partial_y = \sin \phi \partial_r + \frac{\cos \phi}{r} \partial_\phi \quad (32)$$

in the polar coordinate, we are ready to calculate the longitudinal “spin conductivity”

$$\sigma_{sp}^\psi = - \int_0^\infty dr r \int_0^\infty dr' r' \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \lim_{\Omega \rightarrow 0} \frac{\Im \Pi_{xx}^{ss}(r, r', \phi - \phi', \Omega + i\delta)}{\Omega}, \quad (33)$$

A lengthy but straightforward calculation, which is given in appendix C1, yields

$$\begin{aligned} \sigma_{sp}^\psi = & 2\pi^2 t^2 |\mathcal{C}|^4 \sum_{\sigma=\pm} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \int_{-\infty}^\infty d\nu_\sigma \left(-\frac{\partial f(\Omega)}{\partial \Omega} \right)_{\Omega=\nu_\sigma} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}'_\sigma) \sum_n [\mathcal{R}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma)]^2 \\ & + 16\pi^2 \lambda_{so}^2 |\mathcal{C}|^4 \sum_{\sigma=\pm} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \int_{-\infty}^\infty d\nu_\sigma \left(-\frac{\partial f(\Omega)}{\partial \Omega} \right)_{\Omega=\nu_\sigma} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}'_\sigma) \sum_n [\mathcal{S}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma)]^2, \end{aligned} \quad (34)$$

where $\mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma)$ is the spectral function for itinerant electrons,

$$\mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) = -\frac{1}{\pi} \frac{\Im \Sigma_\sigma(\nu_\sigma, \mathcal{E}_\sigma)}{[\nu_\sigma - \Re \Sigma(\nu_\sigma, \mathcal{E}_\sigma) - \mathcal{E}_\sigma]^2 + [\Im \Sigma_\sigma(\nu_\sigma, \mathcal{E}_\sigma)]^2}, \quad (35)$$

and $\mathcal{R}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma)$ and $\mathcal{S}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma)$ are composed of radial wave functions, given by

$$\begin{aligned} \mathcal{R}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) &= \int_0^\infty dr r \left([\partial_r F_\sigma^n(r; \mathcal{E}_\sigma)] F_\sigma^{n+1}(r; \mathcal{E}'_\sigma) - F_\sigma^n(r; \mathcal{E}_\sigma) [\partial_r F_\sigma^{n+1}(r; \mathcal{E}'_\sigma)] \right), \\ \mathcal{S}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) &= \int_0^\infty dr r f(r) F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^{n+1}(r; \mathcal{E}'_\sigma), \end{aligned} \quad (36)$$

respectively. $f(r) = \frac{2\xi r}{\xi^2 + r^2}$ in $\mathcal{S}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma)$ originates from the skyrmion configuration of the spin component. We note that only nearest neighbor angular-momentum channels are coupled. The cross correlation between the

spin-current and spin-density, proportional to $t\lambda_{so}$ in Eq. (22), vanishes for the longitudinal conductivity.

If we consider the non-interacting limit of $\Re \Sigma_\sigma(\nu_\sigma), \Im \Sigma_\sigma(\nu_\sigma) \rightarrow 0$, we can simplify this ex-

pression further, resorting to the asymptotic form of the radial wave function in the case of $\lambda_{so} = 0$. This

procedure is shown in appendix C2, and the final analytic expression is given by

$$\begin{aligned} \sigma_{sp}^\psi(T \rightarrow 0) = & 2\pi^2 t^2 |\mathcal{C}|^4 (C'_1)^4 \xi^{2(1+\sqrt{2})} \sum_n L^{2n+2} \left\{ \frac{{}_2F_1[1+n, -\sqrt{2}, 2+n, -(L/\xi)^2]}{2(n+1)} \right. \\ & \left. + \left(\frac{L}{\xi}\right)^2 \frac{{}_2F_1[2+n, -\sqrt{2}, 3+n, -(L/\xi)^2]}{2(n+2)} \right\}, \end{aligned} \quad (37)$$

where L is the system size and ξ is the skyrmion core size. The hypergeometric function is defined in the Mathematica program.

As discussed before, the electrical conductivity is the same as the spin conductivity

$$\sigma_{el}^\psi = \sigma_{sp}^\psi. \quad (38)$$

The cross longitudinal conductivity is given by

$$\begin{aligned} \sigma_{sp-el}^\psi = & 2\pi^2 t^2 |\mathcal{C}|^4 \sum_{\sigma=\pm} \sigma \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \int_{-\infty}^{\infty} d\nu_\sigma \left(-\frac{\partial f(\Omega)}{\partial \Omega} \right)_{\Omega=\nu_\sigma} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}'_\sigma) \sum_n [\mathcal{R}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma)]^2 \\ & + 16\pi^2 \lambda_{so}^2 |\mathcal{C}|^4 \sum_{\sigma=\pm} \sigma \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \int_{-\infty}^{\infty} d\nu_\sigma \left(-\frac{\partial f(\Omega)}{\partial \Omega} \right)_{\Omega=\nu_\sigma} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}'_\sigma) \sum_n [\mathcal{S}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma)]^2. \end{aligned} \quad (39)$$

As mentioned before, this correlation function is finite because we are considering ferromagnetism. But, this cross effect is not crucial for the skyrmion dynamics, just modifying the transport coefficient.

ward calculation given in appendix D1 leads to the following expression,

D. Hall conductivity

The Hall conductivity can be obtained along the same strategy as the longitudinal conductivity. A straightforward

$$\begin{aligned} \sigma_{sH}^\psi = & 2\pi^2 t^2 |\mathcal{C}|^4 \sum_{\sigma=\pm} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \int_{-\infty}^{\infty} d\nu_\sigma \int_{-\infty}^{\infty} d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{(\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma) \\ & \times \sum_n \left\{ \mathcal{O}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \mathcal{R}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) - \mathcal{P}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \mathcal{Q}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \right\} \\ & + 8\pi^2 t \lambda_{so} |\mathcal{C}|^4 \sum_{\sigma=\pm} \sigma \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \int_{-\infty}^{\infty} d\nu_\sigma \int_{-\infty}^{\infty} d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{(\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma) \sum_n \mathcal{O}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \mathcal{S}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \\ & + 16\pi^2 \lambda_{so}^2 |\mathcal{C}|^4 \sum_{\sigma=\pm} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \int_{-\infty}^{\infty} d\nu_\sigma \int_{-\infty}^{\infty} d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{(\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma) \sum_n [\mathcal{S}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma)]^2, \end{aligned} \quad (40)$$

where

$$\begin{aligned}
\mathcal{O}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) &= \int_0^\infty dr F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^{n+1}(r; \mathcal{E}'_\sigma), & \mathcal{P}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) &= \int_0^\infty dr r F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^{n+1}(r; \mathcal{E}'_\sigma), \\
\mathcal{Q}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) &= \int_0^\infty dr \left([\partial_r F_\sigma^n(r; \mathcal{E}_\sigma)] F_\sigma^{n+1}(r; \mathcal{E}'_\sigma) - F_\sigma^n(r; \mathcal{E}_\sigma) [\partial_r F_\sigma^{n+1}(r; \mathcal{E}'_\sigma)] \right), \\
\mathcal{R}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) &= \int_0^\infty dr r \left([\partial_r F_\sigma^n(r; \mathcal{E}_\sigma)] F_\sigma^{n+1}(r; \mathcal{E}'_\sigma) - F_\sigma^n(r; \mathcal{E}_\sigma) [\partial_r F_\sigma^{n+1}(r; \mathcal{E}'_\sigma)] \right), \\
\mathcal{S}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) &= \int_0^\infty dr r f(r) F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^{n+1}(r; \mathcal{E}'_\sigma).
\end{aligned} \tag{41}$$

The spin-orbit interaction contributes to the Hall conductivity.

This expression reveals that the Hall conductivity does not vanish even without the spin-orbit coupling because the r integration differs from the r' integration. The underlying mechanism is that the chirality is preferred, reflected in the Heun's confluent function. As a result, the r integral becomes different from the r' integral, originating from the angular dependence.

The finite Hall conductivity becomes clearer if one focuses on the asymptotic expression of the radial wave function. We simplify Eq. (40) further in the non-interaction limit, shown in appendix D2. We note that the exponential decay of the asymptotic form is expected

to make the contribution from the radial integral finite for the Hall conductivity. Since this spin Hall conductivity corresponds to $\sigma_{ss}^H = \int dx \int dy \Theta_{ss}(\mathbf{x})$, the finite Hall conductivity means that it is not proportional to L^2 as discussed before.

The charge Hall conductivity is the same as the spin Hall conductivity

$$\sigma_{eH}^\psi = \sigma_{sH}^\psi, \tag{42}$$

implying that σ_{cc}^H is a constant, not proportional to L^2 as discussed in Eq. (12).

The cross Hall coefficient corresponding to σ_{sc}^H in Eq. (8) is given by

$$\begin{aligned}
\sigma_{seH}^\psi &= 2\pi^2 t^2 |\mathcal{C}|^4 \sum_{\sigma=\pm} \sigma \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \int_{-\infty}^\infty d\nu_\sigma \int_{-\infty}^\infty d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{(\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma) \\
&\times \sum_n \left\{ \mathcal{O}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \mathcal{R}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) - \mathcal{P}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \mathcal{Q}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \right\} \\
&+ 8\pi^2 t \lambda_{so} |\mathcal{C}|^4 \sum_{\sigma=\pm} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \int_{-\infty}^\infty d\nu_\sigma \int_{-\infty}^\infty d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{(\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma) \sum_n \mathcal{O}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \mathcal{S}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \\
&+ 16\pi^2 \lambda_{so}^2 |\mathcal{C}|^4 \sum_{\sigma=\pm} \sigma \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \int_{-\infty}^\infty d\nu_\sigma \int_{-\infty}^\infty d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{(\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma) \sum_n [\mathcal{S}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma)]^2.
\end{aligned} \tag{43}$$

This is also non-vanishing because of the $J_H S$ term in the effective action, resulting from the different population between \uparrow and \downarrow itinerant electrons, proportional to $J_H S$.

V. SUMMARY

We investigated dynamics of skyrmions under spin currents driven by electric field in itinerant ferromagnets. We developed a novel framework based on the effective U(1) gauge theory formulation [Eq. (1) and Eq. (2)], where the Maxwell-Chern-Simons equation [Eq. (4) with Eq. (5)] is the key equation for soliton dynamics. Although this framework differs from the Landau-Lifshitz-

Gilbert equation approach, both equations turn out to have essentially the same ingredient as it should be. Indeed, we recovered the expected result of the Landau-Lifshitz-Gilbert equation approach [Eq. (20)] from the Maxwell equation framework [Eq. (7)].

An important improvement beyond the previous study lies in the mutual feedback effect for both skyrmion dynamics and electron motion, where the internal magnetic flux of the skyrmion gives rise to the Hall motion to itinerant electrons or vice versa. This physics can be described by the Chern-Simons terms [Eq. (2)] in the skyrmion moving frame. As a result, we revealed that the skyrmion motion follows not only the electric field but also its transverse direction [Eq. (10)]. An interest-

ing observation is that even if an insulating state is considered, the electric field will induce the dissipationless skyrmion Hall current due to the Chern-Simons terms [Eq. (12)]. In particular, we predict that the dissipationless skyrmion Hall current will survive even in the thermodynamic limit as far as the spin-orbit interaction is introduced, expected to realize in the surface state of three dimensional topological insulators when magnetic impurities are deposited.

We evaluated the Hall conductivity in the presence of the single skyrmion potential, showing that it is non-vanishing indeed although it will vanish in the thermodynamic limit because we are considering only one skyrmion without the spin-orbit interaction. A careful treatment is required because we should construct new eigen basis in the single skyrmion system [Section IV-B]. Based on this construction, we calculated all kinds of correlation functions explicitly, and found the general expression for the Hall conductivity [Eq. (40) with Eq. (41)].

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Appendix A: Mapping from the double exchange model to an effective U(1) gauge theory in the strong coupling limit

We start from the ferromagnetic Kondo lattice model or the double exchange model with the Rashba spin-orbit coupling

$$\begin{aligned} Z &= \int Dc_{i\sigma} D\vec{S}_i \exp\left(-S_B - \int_0^\beta d\tau L\right), \\ L &= \sum_i c_{i\sigma}^\dagger (\partial_\tau - \mu) c_{i\sigma} - t \sum_{ij} (c_{i\sigma}^\dagger c_{j\sigma} + H.c.) \\ &\quad - J_H \sum_i c_{i\alpha}^\dagger (\vec{\sigma} \cdot \vec{S}_i)_{\alpha\beta} c_{i\beta} \\ &\quad - i\lambda_{so} \sum_i \sigma_{\alpha\beta}^{\mathbf{a}} (c_{i+\hat{\mathbf{a}}\alpha}^\dagger c_{i\beta} - c_{i\alpha}^\dagger c_{i+\hat{\mathbf{a}}\beta}), \end{aligned} \quad (\text{A1})$$

where $c_{i\sigma}$ represents the conduction electron field and \vec{S}_i expresses the localized spin. μ is an electron chemical potential and t is the wave-function overlap integral for conduction electrons. J_H is the exchange coupling constant, set to be positive. λ_{so} is the Rashba spin-orbit coupling constant, which breaks the inversion symmetry, realized on the surface or in the inversion-symmetry breaking material. S_B is the single-spin Berry phase term in the spin coherent-state representation, given by

$$S_B = iS \int_0^\beta d\tau \sum_i \partial_\tau \phi_i (1 - \cos \theta_i), \quad (\text{A2})$$

where the spin field is expressed by two angles, $\vec{S}_i = (\sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i)$.

If we consider the system such that dynamics of localized spins is much slower than that of itinerant electrons, spins of conduction electrons follow localized spins. This situation allows us to take the strong coupling approach

$$\begin{aligned} -J_H \vec{S}_i \cdot \vec{\sigma}_{\alpha\beta} &= -J_H S U_{i\alpha\gamma} \sigma_{\gamma\delta}^z U_{i\delta\beta}^\dagger, \\ \psi_{i\alpha} &= U_{i\alpha\beta}^\dagger c_{i\beta}, \end{aligned} \quad (\text{A3})$$

where the unitary matrix field $U_i = \begin{pmatrix} z_{i\uparrow} & z_{i\downarrow}^\dagger \\ z_{i\downarrow} & -z_{i\uparrow}^\dagger \end{pmatrix}$ consists of a bosonic spinon field $z_{i\sigma}$ and the alignment of the itinerant spin to the localized spin introduces an electron field $\psi_{i\alpha}$. The spinon field can be represented in the following way, $z_{i\uparrow} = e^{-i\frac{\phi_i}{2}} \cos \frac{\theta_i}{2}$ and $z_{i\downarrow} = e^{i\frac{\phi_i}{2}} \sin \frac{\theta_i}{2}$, which gives the self-consistent expression $\vec{S}_i = \frac{1}{2} z_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} z_{i\beta}$.

Representing Eq. (A1) for $c_{i\sigma}$ and \vec{S}_i in terms of $\psi_{i\alpha}$ and $U_{i\alpha\beta}$ [Eq. (A3)], and taking the continuum approximation, we reach the following expression

$$\begin{aligned}
Z &= \int D\psi_\alpha DU_{\alpha\beta} \delta(U_{\alpha\gamma}^\dagger U_{\gamma\beta} - \delta_{\alpha\beta}) e^{-S_B - \int_0^\beta d\tau \int d^2r \mathcal{L}}, \\
\mathcal{L} &= \psi_\alpha^\dagger [(\partial_\tau - \mu - 2t)\delta_{\alpha\beta} + U_{\alpha\gamma}^\dagger \partial_\tau U_{\gamma\beta}] \psi_\beta + t \left(\partial_{\mathbf{r}} \psi_\beta^\dagger - \psi_\alpha^\dagger U_{\alpha\gamma}^\dagger (\partial_{\mathbf{r}} U_{\gamma\beta}) \right) \left(\partial_{\mathbf{r}} \psi_\beta - (\partial_{\mathbf{r}} U^\dagger)_{\beta\gamma} U_{\gamma\alpha} \psi_\alpha \right) - J_H S \sigma \psi_\sigma^\dagger \psi_\sigma \\
&+ t \psi_\alpha^\dagger \left(\partial_{\mathbf{r}} U^\dagger + [U^\dagger \partial_{\mathbf{r}} U] U^\dagger \right)_{\alpha\gamma} \left(\partial_{\mathbf{r}} U + U[(\partial_{\mathbf{r}} U^\dagger) U] \right)_{\gamma\beta} \psi_\beta - t \partial_{\mathbf{r}} (\psi_\alpha^\dagger \psi_\alpha) \\
&+ i \lambda_{so} \left(\psi_\gamma^\dagger U_{\gamma\alpha}^\dagger \sigma_{\alpha\beta}^{\mathbf{a}} U_{\beta\delta} (\partial_{\mathbf{a}} \psi_\delta) - (\partial_{\mathbf{a}} \psi_\gamma^\dagger) U_{\gamma\alpha}^\dagger \sigma_{\alpha\beta}^{\mathbf{a}} U_{\beta\delta} \psi_\delta + \psi_\gamma^\dagger U_{\gamma\alpha}^\dagger \sigma_{\alpha\beta}^{\mathbf{a}} U_{\beta\xi} U_{\xi\chi}^\dagger (\partial_{\mathbf{a}} U_{\chi\delta}) \psi_\delta - \psi_\gamma^\dagger (\partial_{\mathbf{a}} U_{\gamma\alpha}^\dagger) U_{\alpha\xi} U_{\xi\chi}^\dagger \sigma_{\chi\beta}^{\mathbf{a}} U_{\beta\delta} \psi_\delta \right).
\end{aligned} \tag{A4}$$

Introducing the Berry gauge field as the spin connection, $\mathcal{A}_{\alpha\beta}^\nu = -i[\{\partial_\nu U^\dagger\}U]_{\alpha\beta}$, we can rewrite the above

as follows

$$\begin{aligned}
Z &= \int D\psi_\alpha DU_{\alpha\beta} D\mathcal{A}_{\alpha\beta}^\nu \delta(U_{\alpha\gamma}^\dagger U_{\gamma\beta} - \delta_{\alpha\beta}) \delta(\mathcal{A}_{\alpha\beta}^\nu + i[\{\partial_\nu U^\dagger\}U]_{\alpha\beta}) \delta(\partial_r \mathcal{A}_{\alpha\beta}^r) e^{-S_B - \int_0^\beta d\tau \int d^2r \mathcal{L}}, \\
\mathcal{L} &= \psi_\alpha^\dagger [(\partial_\tau - \mu - 2t - 2\lambda_{so}^2/t - J_H S \alpha) \delta_{\alpha\beta} - i\mathcal{A}_{\alpha\beta}^r] \psi_\beta \\
&+ t \left(\partial_{\mathbf{r}} \psi_\beta^\dagger + i\psi_\alpha^\dagger [\mathcal{A}_{\alpha\beta}^{\mathbf{r}} + (\lambda_{so}/t) U_{\alpha\gamma}^\dagger \sigma_{\gamma\delta}^{\mathbf{r}} U_{\delta\beta}] \right) \left(\partial_{\mathbf{r}} \psi_\beta - i[\mathcal{A}_{\beta\gamma}^{\mathbf{r}} + (\lambda_{so}/t) U_{\beta\alpha}^\dagger \sigma_{\alpha\delta}^{\mathbf{r}} U_{\delta\gamma}] \psi_\gamma \right) \\
&+ t \psi_\alpha^\dagger \left(\partial_{\mathbf{r}} U_{\alpha\gamma}^\dagger - i[\mathcal{A}_{\alpha\delta}^{\mathbf{r}} + (\lambda_{so}/t) U_{\alpha\beta}^\dagger \sigma_{\beta\chi}^{\mathbf{r}} U_{\chi\delta}] U_{\delta\gamma}^\dagger \right) \left(\partial_{\mathbf{r}} U_{\gamma\beta} + iU_{\gamma\xi} [\mathcal{A}_{\xi\beta}^{\mathbf{r}} + (\lambda_{so}/t) U_{\xi\gamma}^\dagger \sigma_{\gamma\delta}^{\mathbf{r}} U_{\delta\beta}] \right) \psi_\beta.
\end{aligned} \tag{A5}$$

Remember that this effective field theory is just the change of variables in the microscopic model Eq. (A1).

Because this SU(2) gauge theory formulation is quite complicated, we perform the U(1) approximation. One

can understand this procedure as the staggered-flux ansatz in the SU(2) slave-boson theory, where the SU(2) gauge symmetry is reduced to the U(1) symmetry [19]. Then, we find an effective U(1) gauge theory [20]

$$\begin{aligned}
Z &= \int D\psi_\sigma Dz_\sigma Da_\mu \delta(|z_\sigma|^2 - 1) \delta(\partial_r a_r) e^{-S_B - \int_0^\beta d\tau \int d^2r \mathcal{L}}, \quad \mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_z, \\
\mathcal{L}_\psi &= \psi_\sigma^\dagger (\partial_\tau - \mu_r - J_H S \sigma - i\sigma a_\tau - iA_\tau) \psi_\sigma + t [|\partial_{\mathbf{r}} - i\sigma a_{\mathbf{r}} - iA_{\mathbf{r}} - i\sigma(\lambda_{so}/t) z_\alpha^\dagger \sigma_{\alpha\beta}^{\mathbf{r}} z_\beta| \psi_\sigma]^2, \\
\mathcal{L}_z &= \rho_s z_\sigma^\dagger (\partial_\tau - ia_\tau) z_\sigma + t \rho_s [|\partial_{\mathbf{r}} - ia_{\mathbf{r}} - i(\lambda_{so}/t) z_\alpha^\dagger \sigma_{\alpha\beta}^{\mathbf{r}} z_\beta| z_\sigma]^2,
\end{aligned} \tag{A6}$$

where an electromagnetic vector potential $A_{\mathbf{r}}$ is introduced. It is interesting to see that the internal gauge field $a_{\mathbf{r}}$ couples to the spin current of the electron field while the electromagnetic field $A_{\mathbf{r}}$ does to the charge current. $\mu_r = \mu + 2t + 2\lambda_{so}^2/t$ is an effective chemical potential for itinerant electrons. The spinon part is reduced to the CP¹ representation of the ferromagnetic O(3) nonlinear σ model, where the time derivative term is added explicitly. This time derivative term is expected to appear from quantum corrections, i.e., the self-energy correction to the spinon dynamics. In the antiferromagnetic case the second order time derivative term can be applied to. The Rashba spin-orbit coupling gives rise to an interac-

tion term between the spin current and spin, quenching the spin direction to the momentum or current direction.

Considering that the Berry phase term is associated with a background potential, we are allowed to take into account the saddle-point configuration of the gauge field for the Berry phase term

$$a_\mu = -\frac{i}{2} [z_\sigma^\dagger (\partial_\mu z_\sigma) - (\partial_\mu z_\sigma^\dagger) z_\sigma] = -\frac{\partial_\mu \phi}{2} \cos \theta.$$

Then, the Berry phase term can be written as follows

$$S_B = 2iS \int_0^\beta d\tau \int d^2r a_\tau, \tag{A7}$$

where $iS \int_0^\beta d\tau \int d^2r \partial_\tau \phi = 0$ is used for the skyrmion configuration.

Appendix B: Single particle Green's function in the single skyrmion potential

1. Schrodinger equation with the single skyrmion potential

We start from the Schrodinger equation with the single skyrmion potential

$$(-i\partial_t - \mu_r - J_H S\sigma - i\sigma a_r)\psi_\sigma - t(\partial_r - i\sigma a_r - iA_r)^2\psi_\sigma = 0, \quad (\text{B1})$$

where the spin-orbit interaction is neglected. Introducing the polar coordinate associated with the symmetry of the skyrmion potential,

$$-t\left(\hat{r}\partial_r + \hat{\phi}\frac{\partial_\phi}{r} - i\sigma a(r)\hat{\phi}\right) \cdot \left(\hat{r}\partial_r + \hat{\phi}\frac{\partial_\phi}{r} - i\sigma a(r)\hat{\phi}\right) = -t\left(\partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\phi^2}{r^2} - 2i\sigma a(r)\frac{\partial_\phi}{r} - a^2(r)\right), \quad (\text{B2})$$

where the gauge potential $a(r)\hat{\phi} = a_x(x, y)\hat{x} + a_y(x, y)\hat{y}$ in the polar coordinate is given by

$$a(r) = \sqrt{a_x^2 + a_y^2} = \frac{r}{\xi^2 + r^2} \quad (\text{B3})$$

for the single skyrmion solution, we rewrite the above Schrodinger equation as follows

$$(-i\partial_t - \mu_r - J_H S\sigma)\psi_\sigma(r, \phi, t) - t\left\{\partial_r^2 + \frac{\partial_r}{r} + \frac{1}{r^2}\left(\partial_\phi + i\sigma\frac{r^2}{\xi^2 + r^2}\right)^2\right\}\psi_\sigma(r, \phi, t) = 0. \quad (\text{B4})$$

2. Heun's confluent equation

Inserting Eq. (27) with Eq. (28) into Eq. (B4), we obtain the eigen value problem for the radial part

$$\partial_r^2 F_\sigma^n(r) + \frac{\partial_r}{r} F_\sigma^n(r) - \frac{1}{r^2}\left(n + \sigma\frac{r^2}{\xi^2 + r^2}\right)^2 F_\sigma^n(r) + t^{-1}(E_\sigma + \mu_r + J_H S\sigma)F_\sigma^n(r) = 0. \quad (\text{B5})$$

Performing the change of variables $r^2 = -\xi^2 t$ and introducing $F_\sigma^n(t) = t^p(t-1)^q Y_\sigma^n(t)$ with constants p and q , this equation can be written as follows

$$\partial_t^2 Y_\sigma^n(t) + \left(\frac{2p+1}{t} + \frac{2q}{t-1}\right)\partial_t Y_\sigma^n(t) + \left(\frac{p^2 - n^2/4}{t^2} + \frac{(2p+1)q - n\sigma/2}{t(t-1)} + \frac{q(q-1) - 1/4}{(t-1)^2} - \frac{\xi^2}{4t}\mathcal{E}_\sigma\right)Y_\sigma^n(t) = 0, \quad (\text{B6})$$

where $\mathcal{E}_\sigma \equiv t^{-1}(E_\sigma + \mu_r + J_H S\sigma)$ is an effective energy level. Surprisingly, this equation is known to be the Heun's confluent equation, and its solution is well understood as follows

$$Y_\sigma^n(t) = \text{HeunC}\left(0, n, \sqrt{2}, \frac{1+n}{2} - \frac{\xi^2}{4}\mathcal{E}_\sigma, \frac{1-\sigma n}{2} + \frac{\xi^2}{4}\mathcal{E}_\sigma, t\right). \quad (\text{B7})$$

A general expression of the Heun's confluent equation [28] is given by

$$\frac{d^2 Y(z)}{dz^2} - \frac{[-\alpha z^2 + (\alpha - \beta - \gamma - 2)z + \beta + 1]}{z(z-1)} \frac{dY(z)}{dz} - \frac{[(-\beta - \gamma - 2)\alpha - 2\delta]z + (\beta + 1)z + (-\gamma - 1)\beta - \gamma - 2\eta}{2z(z-1)} Y(z) = 0, \quad (\text{B8})$$

satisfying two boundary conditions such as

$$Y(z=0) = 1, \quad \left.\frac{dY(z)}{dz}\right|_{z=0} = \frac{(-\alpha + \gamma + 1)\beta + \gamma - \alpha + 2\eta}{2(\beta + 1)}. \quad (\text{B9})$$

A general solution of this equation is known to be the Heun's confluent function

$$Y(z) = \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; z), \quad (\text{B10})$$

where p and q are determined by

$$p^2 = \frac{n^2}{4}, \quad q(q-1) = \frac{1}{4}, \quad (B11)$$

and all other coefficients are given by p and q in the following way

$$\alpha = 0, \quad \beta = 2p, \quad \gamma = 2q - 1, \quad \delta = p + \frac{1}{2} - \frac{\xi^2}{4}\mathcal{E}_\sigma, \quad \eta = \frac{1 - n\sigma}{2} + \frac{\xi^2}{4}\mathcal{E}_\sigma. \quad (B12)$$

Two independent quantum numbers appear to be an angular momentum n and an energy eigen value \mathcal{E}_σ , where n is an integer while \mathcal{E}_σ turns out to be continuous.

In order to understand time reversal symmetry breaking in the Heun's confluent function, it is valuable to find its asymptotic form in the $r \rightarrow \infty$ limit. The corresponding Schrodinger equation is given by

$$\frac{2p + 2q + 1}{t} \partial_t Y_\sigma^n(t) - \frac{\xi^2}{4t} \mathcal{E}_\sigma Y_\sigma^n(t) \approx 0, \quad (B13)$$

where dominant terms are selected by the $t \rightarrow -\infty$ limit. It is straightforward to solve this differential equation. Introducing $p = n/2$ and $q = (1 + \sqrt{2})/2$ into the solution, we obtain

$$Y_\sigma^n(r \rightarrow \infty) \propto \exp\left(-\frac{\mathcal{E}_\sigma}{4(n+2+\sqrt{2})}r^2\right). \quad (B14)$$

3. A Green's function in a skyrmion background

Inserting Eq. (27) with Eq. (28) into the Green's function, we obtain

$$G_{\sigma\sigma}(r, r', \phi - \phi', t - t') = -|\mathcal{C}|^2 \int d\mathcal{E}_\sigma e^{-i\mathcal{E}_\sigma(t-t')} \sum_n e^{in(\phi-\phi')} F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) \langle a_\sigma(n, \mathcal{E}_\sigma) a_\sigma^\dagger(n, \mathcal{E}_\sigma) \rangle. \quad (B15)$$

We note that the radial coordinate cannot be $r - r'$ due to translational symmetry breaking.

Performing the Fourier transformation for time, we obtain

$$G_{\sigma\sigma}(r, r', \phi - \phi', \omega + i\delta) = |\mathcal{C}|^2 \int d\mathcal{E}_\sigma \sum_n e^{in(\phi-\phi')} F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) \left(\frac{f(\mathcal{E}_\sigma)}{\omega - \mathcal{E}_\sigma - i\delta} + \frac{1 - f(\mathcal{E}_\sigma)}{\omega - \mathcal{E}_\sigma + i\delta} \right) \quad (B16)$$

in the real frequency, where $f(\mathcal{E}_\sigma)$ is the Fermi-Dirac distribution function, while given by

$$G_{\sigma\sigma}(r, r', \phi - \phi', i\omega) = |\mathcal{C}|^2 \int d\mathcal{E}_\sigma \sum_n e^{in(\phi-\phi')} \frac{F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma)}{i\omega - \mathcal{E}_\sigma} \quad (B17)$$

in the Matsubara frequency.

We introduce the self-energy correction, resulting from both elastic impurity scattering and inelastic interaction effects with gauge fluctuations

$$\Sigma_\sigma(i\omega, \mathcal{E}_\sigma) = \Sigma_\sigma^{imp}(i\omega, \mathcal{E}_\sigma) + \Sigma_\sigma^{ele}(i\omega, \mathcal{E}_\sigma). \quad (B18)$$

Then, the most general expression for the single particle Green's function is given by

$$G_{\sigma\sigma}(r, r', \phi - \phi', i\omega) = |\mathcal{C}|^2 \int d\mathcal{E}_\sigma \sum_n e^{in(\phi-\phi')} \frac{F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma)}{i\omega - \Sigma_\sigma(i\omega, \mathcal{E}_\sigma) - \mathcal{E}_\sigma}. \quad (B19)$$

Appendix C: Polarization functions for longitudinal conductivity

1. Formal expressions

For convenience, we decompose Π_{xx}^{ss} into four contributions;

$$\Pi_{xx}^{ss}(r, r', \phi - \phi', i\Omega) = \sum_{K=A,B,C,D} \Pi_{xx}^{ss(K)}(r, r', \phi - \phi', i\Omega). \quad (C1)$$

The first part is the conventional particle-hole channel in the presence of the single skyrmion potential,

$$\begin{aligned} \Pi_{xx}^{ss(A)}(r, r', \phi - \phi', i\Omega) = & -t^2 |\mathcal{C}|^4 \cos \phi \cos \phi' \sum_n \sum_{n'} e^{i(n-n')(\phi-\phi')} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \\ & \left\{ [\partial_r F_\sigma^n(r; \mathcal{E}_\sigma)] F_\sigma^n(r'; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^{n'}(r'; \mathcal{E}'_\sigma)] F_\sigma^{n'}(r; \mathcal{E}'_\sigma) - [\partial_r F_\sigma^n(r; \mathcal{E}_\sigma)] [\partial_{r'} F_\sigma^n(r'; \mathcal{E}_\sigma)] F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \right. \\ & \left. - F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^{n'}(r'; \mathcal{E}'_\sigma)] [\partial_r F_\sigma^{n'}(r; \mathcal{E}'_\sigma)] + F_\sigma^n(r; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^n(r'; \mathcal{E}_\sigma)] F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) [\partial_r F_\sigma^{n'}(r; \mathcal{E}'_\sigma)] \right\} \\ & \int_{-\infty}^{\infty} d\nu_\sigma \int_{-\infty}^{\infty} d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{i\Omega - (\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma), \end{aligned} \quad (C2)$$

where the spectral function for itinerant electrons is given by Eq. (35). In the ideal non-interacting limit $\Re \Sigma_\sigma(\nu_\sigma), \Im \Sigma_\sigma(\nu_\sigma) \rightarrow 0$ we obtain

$$\int_{-\infty}^{\infty} d\nu_\sigma \int_{-\infty}^{\infty} d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{i\Omega - (\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma) \rightarrow \frac{f(\mathcal{E}_\sigma) - f(\mathcal{E}'_\sigma)}{i\Omega - (\mathcal{E}_\sigma - \mathcal{E}'_\sigma)}, \quad (C3)$$

nothing but the particle-hole polarization function in the Fermi gas.

Other pieces are given by

$$\begin{aligned} \Pi_{xx}^{ss(B)}(r, r', \phi - \phi', i\Omega) = & t^2 |\mathcal{C}|^4 \frac{\sin \phi}{r} \cos \phi' \sum_n \sum_{n'} i(n - n') e^{i(n-n')(\phi-\phi')} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \\ & \left\{ F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^{n'}(r'; \mathcal{E}'_\sigma)] F_\sigma^{n'}(r; \mathcal{E}'_\sigma) - F_\sigma^n(r; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^n(r'; \mathcal{E}_\sigma)] F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \right. \\ & \left. - F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) [\partial_r F_\sigma^{n'}(r; \mathcal{E}'_\sigma)] + F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) [\partial_r F_\sigma^{n'}(r; \mathcal{E}'_\sigma)] \right\} \\ & \int_{-\infty}^{\infty} d\nu_\sigma \int_{-\infty}^{\infty} d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{i\Omega - (\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma), \end{aligned} \quad (C4)$$

$$\begin{aligned} \Pi_{xx}^{ss(C)}(r, r', \phi - \phi', i\Omega) = & -t^2 |\mathcal{C}|^4 \cos \phi \frac{\sin \phi'}{r'} \sum_n \sum_{n'} i(n - n') e^{i(n-n')(\phi-\phi')} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \\ & \left\{ [\partial_r F_\sigma^n(r; \mathcal{E}_\sigma)] F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) - [\partial_r F_\sigma^n(r; \mathcal{E}_\sigma)] F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \right. \\ & \left. - F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^{n'}(r'; \mathcal{E}'_\sigma)] F_\sigma^{n'}(r; \mathcal{E}'_\sigma) + F_\sigma^n(r; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^n(r'; \mathcal{E}_\sigma)] F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \right\} \\ & \int_{-\infty}^{\infty} d\nu_\sigma \int_{-\infty}^{\infty} d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{i\Omega - (\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma), \end{aligned} \quad (C5)$$

and

$$\begin{aligned} \Pi_{xx}^{ss(D)}(r, r', \phi - \phi', i\Omega) = & -t^2 |\mathcal{C}|^4 \frac{\sin \phi}{r} \frac{\sin \phi'}{r'} \sum_n \sum_{n'} (n - n')^2 e^{i(n-n')(\phi-\phi')} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \\ & \left\{ F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) - F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \right. \\ & \left. - F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) + F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \right\} \\ & \int_{-\infty}^{\infty} d\nu_\sigma \int_{-\infty}^{\infty} d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{i\Omega - (\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma). \end{aligned} \quad (C6)$$

Substituting the above expressions into Eq. (33), it is found that the contributions from $\Pi_{xx}^{ss(D)}(r, r', \phi - \phi', i\Omega)$ and $\Pi_{xx}^{ss(B)}(r, r', \phi - \phi', i\Omega) + \Pi_{xx}^{ss(C)}(r, r', \phi - \phi', i\Omega)$ all vanish. Therefore, we obtain Eq. (34).

2. Asymptotic form

The conductivity is simplified as follows in the non-interacting limit without the spin-orbit interaction

$$\sigma_{sp}^\psi = 2\pi^2 t^2 |\mathcal{C}|^4 \int d\mathcal{E}_\sigma \left(-\frac{\partial f(\Omega)}{\partial \Omega} \right)_{\Omega=\mathcal{E}_\sigma} \sum_n [\mathcal{R}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}_\sigma)]^2. \quad (C7)$$

Resorting to the asymptotic expression, we see

$$\partial_r F_\sigma^n(r \rightarrow \infty; \mathcal{E}_\sigma) = \left((1 + \sqrt{2}) \frac{r}{\xi^2 + r^2} + \frac{n}{r} - \frac{\mathcal{E}_\sigma}{2(n + 2 + \sqrt{2})} r \right) F_\sigma^n(r \rightarrow \infty; \mathcal{E}_\sigma). \quad (\text{C8})$$

Then, we obtain

$$\begin{aligned} \mathcal{R}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}_\sigma) &= - \int_0^\infty dr r \left(\frac{1}{r} + \frac{\mathcal{E}_\sigma}{2(n + 2 + \sqrt{2})(n + 3 + \sqrt{2})} r \right) F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^{n+1}(r; \mathcal{E}_\sigma) \\ &= -(C_1')^2 \int_0^\infty dr r \left(\frac{1}{r} + \frac{\mathcal{E}_\sigma}{2(n + 2 + \sqrt{2})(n + 3 + \sqrt{2})} r \right) (\xi^2 + r^2)^{1+\sqrt{2}} r^{2n+1} \exp\left(-\mathcal{E}_\sigma \frac{2n + 5 + 2\sqrt{2}}{4(n + 2 + \sqrt{2})(n + 3 + \sqrt{2})} r^2\right). \end{aligned} \quad (\text{C9})$$

Inserting this expression into Eq. (C7), we reach the following expression in the $T \rightarrow 0$ limit

$$\begin{aligned} \sigma_{sp}^\psi(T \rightarrow 0) &= 2\pi^2 t^2 |\mathcal{C}|^4 (C_1')^4 \int d\mathcal{E}_\sigma \delta(\mathcal{E}_\sigma) \sum_n \left\{ \int_0^L dr r \left(\frac{1}{r} + \frac{\mathcal{E}_\sigma}{2(n + 2 + \sqrt{2})(n + 3 + \sqrt{2})} r \right) \right. \\ &\quad \times (\xi^2 + r^2)^{1+\sqrt{2}} r^{2n+1} \exp\left(-\mathcal{E}_\sigma \frac{2n + 5 + 2\sqrt{2}}{4(n + 2 + \sqrt{2})(n + 3 + \sqrt{2})} r^2\right) \left. \right\}^2 \\ &= 2\pi^2 t^2 |\mathcal{C}|^4 (C_1')^4 \sum_n \xi^{4+2\sqrt{2}+2n} \left\{ \int_0^{L/\xi} dx x^{2n+1} (1 + x^2)^{1+\sqrt{2}} \right\} \\ &= 2\pi^2 t^2 |\mathcal{C}|^4 (C_1')^4 \sum_n \xi^{4+2\sqrt{2}+2n} \left(\frac{L}{\xi} \right)^{2n+2} \left\{ \frac{{}_2F_1[1 + n, -\sqrt{2}, 2 + n, -(L/\xi)^2]}{2(n + 1)} \right. \\ &\quad \left. + \left(\frac{L}{\xi} \right)^2 \frac{{}_2F_1[2 + n, -\sqrt{2}, 3 + n, -(L/\xi)^2]}{2(n + 2)} \right\}, \end{aligned} \quad (\text{C10})$$

where L/ξ is the ratio between the system size and the skyrmion core size and ${}_2F_1$ is the hypergeometric function.

Appendix D: Polarization functions for Hall conductivity

1. Formal expressions

It is convenient to decompose Π_{xy}^{ss} into four contributions;

$$\Pi_{xy}^{ss}(r, r', \phi - \phi', i\Omega) = \sum_{K=A,B,C,D} \Pi_{xy}^{ss(K)}(r, r', \phi - \phi', i\Omega). \quad (\text{D1})$$

The four polarization functions are given by

$$\begin{aligned} \Pi_{xy}^{ss(A)}(r, r', \phi - \phi', i\Omega) &= -t^2 |\mathcal{C}|^4 \cos \phi \sin \phi' \sum_n \sum_{n'} e^{i(n-n')(\phi-\phi')} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \\ &\quad \left\{ [\partial_r F_\sigma^n(r; \mathcal{E}_\sigma)] F_\sigma^n(r'; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^{n'}(r'; \mathcal{E}'_\sigma)] F_\sigma^{n'}(r; \mathcal{E}'_\sigma) - [\partial_r F_\sigma^n(r; \mathcal{E}_\sigma)] [\partial_{r'} F_\sigma^n(r'; \mathcal{E}_\sigma)] F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \right. \\ &\quad \left. - F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^{n'}(r'; \mathcal{E}'_\sigma)] [\partial_r F_\sigma^{n'}(r; \mathcal{E}'_\sigma)] + F_\sigma^n(r; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^n(r'; \mathcal{E}_\sigma)] F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) [\partial_r F_\sigma^{n'}(r; \mathcal{E}'_\sigma)] \right\} \\ &\quad \int_{-\infty}^\infty d\nu_\sigma \int_{-\infty}^\infty d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{i\Omega - (\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma), \end{aligned} \quad (\text{D2})$$

$$\begin{aligned} \Pi_{xy}^{ss(B)}(r, r', \phi - \phi', i\Omega) &= t^2 |\mathcal{C}|^4 \frac{\sin \phi}{r} \sin \phi' \sum_n \sum_{n'} i(n - n') e^{i(n-n')(\phi-\phi')} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \\ &\quad \left\{ F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^{n'}(r'; \mathcal{E}'_\sigma)] F_\sigma^{n'}(r; \mathcal{E}'_\sigma) - F_\sigma^n(r; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^n(r'; \mathcal{E}_\sigma)] F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \right. \\ &\quad \left. - F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) [\partial_r F_\sigma^{n'}(r; \mathcal{E}'_\sigma)] + F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) [\partial_r F_\sigma^{n'}(r; \mathcal{E}'_\sigma)] \right\} \\ &\quad \int_{-\infty}^\infty d\nu_\sigma \int_{-\infty}^\infty d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{i\Omega - (\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma), \end{aligned} \quad (\text{D3})$$

$$\begin{aligned}
\Pi_{xy}^{ss(C)}(r, r', \phi - \phi', i\Omega) &= t^2 |\mathcal{C}|^4 \cos \phi \frac{\cos \phi'}{r'} \sum_n \sum_{n'} i(n - n') e^{i(n - n')(\phi - \phi')} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \\
&\left\{ [\partial_r F_\sigma^n(r; \mathcal{E}_\sigma)] F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) - [\partial_r F_\sigma^n(r; \mathcal{E}_\sigma)] F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \right. \\
&- F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^{n'}(r'; \mathcal{E}'_\sigma)] F_\sigma^{n'}(r; \mathcal{E}'_\sigma) + F_\sigma^n(r; \mathcal{E}_\sigma) [\partial_{r'} F_\sigma^{n'}(r'; \mathcal{E}'_\sigma)] F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \Big\} \\
&\int_{-\infty}^{\infty} d\nu_\sigma \int_{-\infty}^{\infty} d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{i\Omega - (\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma),
\end{aligned} \tag{D4}$$

and

$$\begin{aligned}
\Pi_{xy}^{ss(D)}(r, r', \phi - \phi', i\Omega) &= t^2 |\mathcal{C}|^4 \frac{\sin \phi}{r} \frac{\cos \phi'}{r'} \sum_n \sum_{n'} (n - n')^2 e^{i(n - n')(\phi - \phi')} \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \\
&\left\{ F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) - F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \right. \\
&- F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) + F_\sigma^n(r; \mathcal{E}_\sigma) F_\sigma^n(r'; \mathcal{E}_\sigma) F_\sigma^{n'}(r'; \mathcal{E}'_\sigma) F_\sigma^{n'}(r; \mathcal{E}'_\sigma) \Big\} \\
&\int_{-\infty}^{\infty} d\nu_\sigma \int_{-\infty}^{\infty} d\nu'_\sigma \frac{f(\nu_\sigma) - f(\nu'_\sigma)}{i\Omega - (\nu_\sigma - \nu'_\sigma)} \mathcal{A}_\sigma(\nu_\sigma, \mathcal{E}_\sigma) \mathcal{A}_\sigma(\nu'_\sigma, \mathcal{E}'_\sigma).
\end{aligned} \tag{D5}$$

It is found that $\Pi_{xy}^{ss(A)}(r, r', \phi - \phi', i\Omega)$ and $\Pi_{xy}^{ss(D)}(r, r', \phi - \phi', i\Omega)$ do not contribute to σ_{sH}^ψ , while $\Pi_{xy}^{ss(B)}(r, r', \phi - \phi', i\Omega)$ and $\Pi_{xy}^{ss(C)}(r, r', \phi - \phi', i\Omega)$ give finite contributions.

2. Asymptotic forms

In the non-interacting limit the Hall conductivity is given by

$$\sigma_{sH}^\psi = 2\pi^2 t^2 |\mathcal{C}|^4 \int d\mathcal{E}_\sigma \int d\mathcal{E}'_\sigma \frac{f(\mathcal{E}_\sigma) - f(\mathcal{E}'_\sigma)}{(\mathcal{E}_\sigma - \mathcal{E}'_\sigma)} \sum_n \left\{ \mathcal{O}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \mathcal{R}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) - \mathcal{P}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \mathcal{Q}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) \right\}, \tag{D6}$$

where the spin-orbit interaction is not introduced.

Inserting the asymptotic expression of the radial wave function as performed in appendix C2, we obtain

$$\begin{aligned}
\mathcal{O}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) &= (\mathcal{C}'_1)^2 \int_0^\infty dr (\xi^2 + r^2)^{1+\sqrt{2}} r^{2n+1} \exp\left(-\left\{\frac{\mathcal{E}_\sigma}{4(n+2+\sqrt{2})} + \frac{\mathcal{E}'_\sigma}{4(n+3+\sqrt{2})}\right\}r^2\right), \\
\mathcal{P}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) &= (\mathcal{C}'_1)^2 \int_0^\infty dr r (\xi^2 + r^2)^{1+\sqrt{2}} r^{2n+1} \exp\left(-\left\{\frac{\mathcal{E}_\sigma}{4(n+2+\sqrt{2})} + \frac{\mathcal{E}'_\sigma}{4(n+3+\sqrt{2})}\right\}r^2\right), \\
\mathcal{Q}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) &= -(\mathcal{C}'_1)^2 \int_0^\infty dr \left(\frac{1}{r} + \frac{\mathcal{E}_\sigma}{2(n+2+\sqrt{2})}r - \frac{\mathcal{E}'_\sigma}{2(n+3+\sqrt{2})}r\right) \\
&\quad \times (\xi^2 + r^2)^{1+\sqrt{2}} r^{2n+1} \exp\left(-\left\{\frac{\mathcal{E}_\sigma}{4(n+2+\sqrt{2})} + \frac{\mathcal{E}'_\sigma}{4(n+3+\sqrt{2})}\right\}r^2\right), \\
\mathcal{R}_\sigma^n(\mathcal{E}_\sigma; \mathcal{E}'_\sigma) &= -(\mathcal{C}'_1)^2 \int_0^\infty dr r \left(\frac{1}{r} + \frac{\mathcal{E}_\sigma}{2(n+2+\sqrt{2})}r - \frac{\mathcal{E}'_\sigma}{2(n+3+\sqrt{2})}r\right) \\
&\quad \times (\xi^2 + r^2)^{1+\sqrt{2}} r^{2n+1} \exp\left(-\left\{\frac{\mathcal{E}_\sigma}{4(n+2+\sqrt{2})} + \frac{\mathcal{E}'_\sigma}{4(n+3+\sqrt{2})}\right\}r^2\right).
\end{aligned} \tag{D7}$$

It is difficult to perform further simplification analytically for the general case of n . However, it is clear that this expression does not vanish due to the factor of r in the r integration.

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